MAS 3105 June 18, 2009
Final Exam and Key Prof. S. Hudson

1a) [each part of problem 1 is 5 points]. In the Rabbit example, we studied $L : R^2 \to R^2$ where $L([a, b]^T) = [a + b, 2a]^T$. Find the matrix representation A of L (w.r.t. the std basis of R^2).

1b) We found two eigenvectors of A, $\mathbf{x_1} = [1, 1]^T$ and $\mathbf{x_2} = [1, -2]^T$, which form a basis X of R^2. Find the corresponding eigenvalues.

1c) Find the matrix representation B of L w.r.t. the basis X.

1d) Find an eigenvector for B. Is it also an eigenvector of A ? Explain briefly.

1e) Let $\mathbf{x} = [2, 3]_X^T$ (the coordinates are wrt the eigenvector basis, X). Find $[L(\mathbf{x}]_S$, in standard coordinates.

2) [20 pts] True-False. You can assume the matrices are all square.

A^TA and AA^T always have the same rank.

If A and B are unitary then AB is unitary.

If A is singular then AB is singular.

If $A^H = -A$ then A is normal.

If A is unitary then A is not defective.

If A is Hermitian and unitary then $A^2 = I$.

If U, V are subspaces of R^n, and $U \perp V$ then $V \subset U^\perp$.

If A is unitary and $\mathbf{x} \in C^n$ then $||A\mathbf{x}|| = ||\mathbf{x}||$.

If λ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.

If λ is an eigenvalue of a unitary matrix then λ is real.

3) [10pts] Suppose A is a 4x4 matrix and det $A = 3$. Find det $(\text{adj}(A))$.

4) [10pts] Choose ONE of these to prove.

a) If an nxn matrix A is diagonalizable, then it has n L.I. eigenvectors.

b) State and prove the Spectral Theorem.

5) [15pts] You are given this $A = QR$ factorization to help with the questions below. Note that the $\frac{1}{2}$ scalar is part of Q.

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

5a) Find a Least Squares solution to the system $A\mathbf{x} = \mathbf{b} = [1,2,3,4]^T$

5b) Find $\mathbf{p} = \text{proj}_{R(A)}\mathbf{b}$.

5c) Let $S = \text{span }\{\mathbf{q_1}, \mathbf{q_2}\} \subset R(Q) \subset R^4$. Find $\text{proj}_S\mathbf{b}$.

6a) [10pts] Diagonalize A. The eigenvalues are ± 1:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6b) [5pts] Find a matrix B so that $B^2 = A$. This may involve your answer to 4a) and may involve complex numbers. If your work results in a product or sum of matrices, you do not have to simplify (but this might help you check your answer).

6c) [5pts] Find e^A (ditto remarks in 4b).

Remarks and Answers: Nobody handed this in early, so I guess it was a bit too long for an 80 minute time period. On the other hand, many people wasted time on calculations they didn't really need (eg, in problem 1, or finding R^{-1} in problem 5).

The average grade was about 55 / 100. The worst results were on problem 5 (especially part c) and 3. You can adjust the usual scale about 10 points downwards for this exam. I have not yet calculated the scale for the semester grades.

1) The first 3-4 answers are in your lecture notes and do not require any real work.

a) A=[1 1;2 0] (in MATLAB notation).

b) 2 and -1 (if you forgot these numbers, just multiply Ax_1 to find λ_1, etc.)

c) B = D = [2 0; 0 -1].

d) Since B is diagonal, it has eigenvectors $[1,0]^T$ and $[0,1]^T$, which are not the same as the eigenvectors of A, at least on the surface. But the coordinates in these vectors are wrt the basis X, so the first one is actually $\mathbf{x_1}$, so it *is* an eigenvector of A in disguise (same idea for the second one). Since this was rather tricky, I gave as much partial credit as possible.

e) $[1,10]^T$. You can multiply by $B = D$ to do L, and then by X (the transition matrix you get from the basis X) to convert to the std basis. Or, you can multiply by X first and then by A. These give the same result since $AX = XD$ (see Ch 6.3).

2) TTTTT TTFTF

3) $3^{4-1} = 27$. If you forgot the 4, recall the HW about det (αA). Also, since this didn't ask for a proof, you could work out an example. You could choose A to be a diagonal matrix with nonzero entries 1,1,1,3. Then adj A is pretty easy to compute (diag with entries 3,3,3,1).

4) See text. Most people chose b) and did it pretty well. Some answers to a) were not well-organized; you should start from $A = XDX^{-1}$, and explain why X contains n LI evecs.

5a) $x = [2.9, -0.1, -0.25]^T$. Start from $Rx = Q^T b$, a simple system that's ready for back substitution. This is one of the reasons for doing a QR factorization. Several people converted this to $x = R^{-1}Q^T b$, which takes much much longer.

5b) Use the answer to 5a), $p = Ax = [3, 2, 3, 2]^T$.

5c) $[5, 5, 5, 5]^T$. This is not very related to the previous parts, but it does use the fact that the two columns of Q must be onl. Based on the theorem in Ch 5.5, add the projections onto q_1 and q_2. By coincidence, $b \perp q_2$, so you could just project onto q_1 (but explain why).

Remark: It's possible to do 5c) like 5b), or to do 5b) like 5c), but I think the explanations above are the simplest.

6a)

$$A = XDX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

This is a routine Ch 6.3 problem (though many people missed it). The given eigenvalues go into D; the eigenvectors into X. For example, $\mathbf{x_1} = [1, 1]^T$ comes from $N(A - I)$. Other answers are possible.

6b) $B = A^{1/2} = XD^{1/2}X^{-1}$ where

$$D^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

The notation $D^{1/2}$ conveys the right idea, but it is not technically correct, since there are many matrices that work as well as the one I gave (eg, you can replace the 1 by -1, or the i by $-i$). There was a HW like this, but expressed more accurately (see text).

6c) $e^A = Xe^D X^{-1}$ where

$$e^D = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$$

which you did not have to simplify further.