MAA 3200, Fall 2010, Key to HW on Shilov Ch 3

An answer key to part of HW5.

1) Define $\rho : R^2 \times R^2 \to R$ by $\rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Each part of this problem below will use this metric instead of the usual one.

1a) Show that ρ is a metric (from the definition).

Answer / Hint: Most people got this.

1b) Draw a picture of $B_2(0) = \{x \in M = R^2 : \rho(x, 0) < 2\}$. This is standard notation for a ball of radius 2 centered at 0 in a metric space [but maybe it is not in the Shilov book?]. Also, I am using "0" as an abbreviation for the origin in R^2, so $0 = (0, 0)$.

Answer / Hint: It is a diamond shape, with vertices at (0,2), (2,0), (0,-2) and (-2,0).

1c) Prove that $P(1, 0)$ is an interior point of $B_2(0)$ using the definition of interior point (find a suitable radius and prove it works).

Answer / Hint: You can show that $B_1(1, 0) \subset B_2(0)$ by picking an arbitrary point, and doing a little algebra (tri ineq).

2) Give an example of a collection of open sets in R^2 such that the intersection of the collection is the closed unit ball. You can use the standard metric, or the one from Problem 1, whichever you prefer.

Answer / Hint: $B_{1+1/n}(0)$ shrinks to the closed unit ball.

3a) Let $f(x, y) = xy$ on R^2. Show that the set $U = \{(x, y) \in R^2 : f(x, y) > 0\}$ is open. You can use the standard metric in this one. Use the defn of open.

Answer / Hint: It might be fastest to do a) and b) together. Let $A = \{(x, y) : x > 0 \land y > 0\} \subset U$. Given $(x, y) \in A$, set $\epsilon =$min $\{x, y\}$, and show that $B_\epsilon(x, y) \subset A$, so that (x, y) is interior. I leave B to you.

3b) Show that U can be partitioned into 2 components, A and B; this means A and B are both open sets, and are disjoint and $A \cup B = U$.

4) Suppose that $x_n \to x \in R^2$, as defined in Section 3.31, using the standard metric in R^2. Show that $x_n \to x$ is also true if we use the metric in Problem 1.

Answer / Hint: Call these metrics ρ_S and ρ_1. Show that for all pairs of points in $x, y \in R^2$, $\rho_1(x, y) \leq 2\rho_S(x, y)$ [fairly simple algebra]. Then compare the definitions of limit using the two metrics [the idea is that if two points are close together in one metric, they are also close in the other metric].