MAA 3200, HW on fields

See Shilov's book for the definition of *field* and *ordered field.* I will not ask you to memorize these definitions.

1) Suppose F is a field and $x, y \in F$ and $xy = 0$. Prove that $x = 0$ or $y = 0$. [Note: if $x \in F$ and $x \neq 0$, then $\exists a \in F, ax = xa = 1$].

2) Recall that $a \equiv b \pmod 5$ is an eq. rel. with 5 eq classes, denoted $Z/\equiv\ = \{[0], [1], [2], [3], [4]\}$. More common notation is $Z_5 = \{0, 1, 2, 3, 4\}$. Define $+ : Z_5 \times Z_5 \to Z_5$ by $[x] + [y] = [x + y]$. For example, 4+3=2 (since $[4] + [3] = [7] = [2]$).

2a) Prove that + is well-defined. (Assume that $[a'] = [a]$ and $[b'] = [b]$. ETS: $[a'] + [b'] = [a] + [b]$).

2b) State a definition of multiplication on Z_5 [similar to the one for addition above] and compute 3 times 3. Show that 3 has a multiplicative inverse (find $x \in Z_5$ such that $3x = 1$). Check for yourself that Z_5 is a field (you do not have to write it out).

2c) Explain why Z_6 is not a field.

2d) Which of these are fields; Z_2 ? Z_3 ? Z_4 ? Can you find a pattern to say when Z_n is a field ?

3) Prove that it is not possible to define $<$ on the complex numbers, to make it an ordered field. [Assume it is. Use the fact that $\exists i \in C$, $i^2 = -1$, and trichotomy, to get a contradiction.] Do you think Z_5 can be made into an ordered field? Explain [Is there an element like $i \in Z_5$?].

Review Problems Related to N

4) Find these elements of N (easy!):

a) $\sigma(4)$
b) $s_3(2)$
c) $s_3(\sigma(2))$ and $\sigma(s_3(2))$ - are they equal?

5) Use the definition of $<$ in N to show that $3 < 10$. You can assume that 1, 3, 4, 7, 9 and 10 are in N and can use known formulas like $1 + 3 = 4$.