# NOTES ON THE THEORY OF INTEREST

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## 0. Introduction

These notes present the Fisherian theory of interest [Fisher (1930)] by means of a one-good, two-period model of production and exchange in an environment of certainty. We draw upon the model developed by Burmeister (1980, Section 2.4) but with a sequential market structure [along the lines of Magill and Quinzii (1996)]. The overall approach is inspired by the pioneering work of Hirshleifer (1970) and the excellent treatise of Cochrane (2001).

We begin with budget constraints and the consumption/saving decision in a pure exchange environment—an environment in which the purchase of debt instruments is the only opportunity for investment. Production and real investment opportunities are added next, followed by the important result known as the Fisher separation theorem, which is considered as the theoretical foundation for the separation of ownership and control in modern corporations. In turn, we move from individual to aggregate behavior and the study of the determination of the interest rate in the full model. We conclude with the study of equilibrium and comparative statics with the aid of the device of 'representative agent'.

In models that study the allocation of resources over time, one has to spell out how the passage of time is represented. We assume throughout that the passage of time is not continuous but a succession of equidistant discrete points $t = 0,1,2,...$ Points in time (or moments of time) are also referred to as *dates*. A *period* is the time interval between two adjacent dates, with the first period being the time from 0 to 1. In other words, the period extended from time $t$ to time $t$+1 is labeled as the $(t+1)th$ period. Further, we suppose that decisions are taken, contracts are set, payments and deliveries are made, and markets operate only at the start of each period, namely at dates $0,1,2,...$ Accordingly, production or consumption decisions established when markets close at time $t$ are carried out in the course of the $(t+1)th$ period; in a similar fashion, decision about stocks of items such as quantities of assets at time $t$ are volumes of assets to be held during the $(t+1)th$ period—the asset holdings of agents when markets reopen at date $t+1$. When we speak of output, real income or consumption at date $t$, we mean cumulative respective flows within the $(t+1)th$ period. Cumulating flows in that fashion is like asserting that the intraperiod distribution of these flows is economically irrelevant[1]. A period may be interpreted as a week, a month, a year, or any other length of calendar time, depending on the context. If the length of each period is taken to be $\Delta t > 0$ instead of 1, the continuity of time can be approximated by taking $\Delta t$ sufficiently small. In principle, the continuous-time version of a discrete-time model is obtained by taking the limit as $\Delta t$ approaches zero.

The Fisherian model we are about to develop is essentially a single-period or two-date model. It involves a prescribed set of decision-makers called *agents*, often identified as natural individuals, who function as consumers, traders, investors, and producers. They make decisions now (at date 0) with an eye to desired outcomes in the future or one period from now, that is at date 1. The absence of concern as to what happens beyond date 1 is captured by saying that agents live for only two periods. At time 0, they know they will die at the end of the second period. While this might seem as a two-period setting, it will become evident that analytically only dates 0 and 1 matter—the designation 'two-date model' would be more appropriate.

Individual agents are characterized by their preferences over alternative consumption plans (i.e. plans about current and future consumption), technological know-how (production function), and endowed income streams. There are perfectly competitive spot markets for goods at dates 0 and 1, and a perfectly competitive bond market at date 0. A bond is a debt instrument issued at date zero and promising to pay one unit of account at date 1.There is a single non-storable good (often referred to as corn) which may serve both as a consumption good and production input (seed or physical capital). The single-good assumption is tantamount to freezing the relative prices of goods at each date; the quantity of corn may be interpreted as a quantity index of goods and its price as the pertinent price level. It is assumed that the price of corn (i.e. the price level of goods) at date 1 is correctly anticipated at date 0. The spot prices of corn at dates 0 and 1 and the price of bonds (at date zero) are denominated in arbitrary units of account referred to as dollars. Not unexpectedly, it turns out that only relative prices matter—so the spot price of date 0 corn is often selected as the *numeraire*. Nevertheless, we prefer to begin by expressing prices and budget constraints in dollar terms along the lines of Barro (2008) so as to show explicitly the transition to real accounts.

## 1. Intertemporal Utility and Budget Constraints; Nominal and Real Interest Rates

### 1.1. Intertemporal Utility

Consider first a **pure exchange economy** (i.e. an economy without production activities) consisting of a set of $N$ agents, with $N$ being a given positive integer. Each agent is specified by (i) an *intertemporal* or *lifetime utility function* $U = U(c_0, c_1)$, representing her preferences over alternative *consumption plans* or *consumption streams* or *time paths of consumption* $(c_0, c_1)$; and (ii) by her *time-distributed endowment of goods* or *endowed output (real income) stream* or simply *endowment* $\boldsymbol{\omega} := (\omega_0, \omega_1)$ [with both entries being nonnegative, $\omega_0 \geq 0$, $\omega_1 \geq 0$, and at

[1] See, for example, Fama and Miller (1972, Ch. 1) and Hirshleifer (1970, Ch. 2) for more discussion on the formal description of the passage of time.

least one of them being strictly positive]. We can imagine that each agent owns a farm (or other production unit), for which all production decisions have been made in advance (i.e. before date 0) and from which a sequence of harvests is forthcoming at the beginning of each period. In a world of certainty, the agent knows not only the current output $\omega_0$ but she correctly anticipates the future harvest $\omega_1$. The symbol $c_t$ stands for consumption at date *t*. Thus $c_0$ denotes present consumption (with the understanding that the quantity of corn $c_0$ is purchased at date 0 for the agent's consumption needs over the first period) and $c_1$ denotes future consumption (with the quantity of corn $c_1$ being obtained at date 1 and consumed over the second period).

Recall that an *indifference curve* in the $c_0c_1$ plane is the locus of all consumption plans that yield the same level of utility (i.e. all the consumption streams that are equally good in the agent's preference scale). The *marginal rate of substitution* $MRS_{01}(c_0, c_1)$ *of* consumption at $t = 0$ *for* consumption at $t = 1$ at a positive consumption stream $(c_0, c_1)$ is related to marginal utilities by

$$MRS_{01}(c_0, c_1) := MU_0(c_0, c_1) / MU_1(c_0, c_1), \tag{1.1}$$

where

$$MU_0 := \partial U / \partial c_0 \quad \text{and} \quad MU_1 := \partial U / \partial c_1.$$

The slope of the agent's indifference curve at $(c_0, c_1)$ is precisely[2] $-MRS_{01}(c_0, c_1)$. Accordingly, the marginal rate of substitution $MRS_{01}(c_0, c_1)$ tells us the maximum amount of future consumption that the agent is willing to forego for a one-unit of marginal increase in current consumption (or the amount of future consumption that the agent must be given to compensate her for a one-unit of marginal decrease in current consumption). In other words, $MRS_{01}(c_0, c_1)$ is the agent's marginal willingness to pay for current consumption, measured in terms of future consumption—a 'subjective relative price'.

It should be recalled that the theory of choice is based on a given preference scale (ranking of alternative consumption plans) rather than on a given function defining use-value in the sense of classical political economy. By definition, a utility function $U(\cdot, \cdot)$ assigns a unique number $U(c_0, c_1)$ to each consumption plan $(c_0, c_1)$ in such a way that for all consumption plans $(c_0', c_1')$ and $(c_0'', c_1'')$, $(c_0', c_1')$ is at least as good as $(c_0'', c_1'')$ in the preference scale of the agent if and only if $U(c_0', c_1') \geq U(c_0'', c_1'')$. It in this precise sense that the intertemporal utility function $U(\cdot, \cdot)$ is said to *represent* the agent's preferences over alternative consumption plans (or that the agent's preferences have a *utility representation*). This is of course an *ordinal utility* concept: if a utility function $U(\cdot, \cdot)$ represents the agent's preferences, then any monotonic

[2] Varian (2006) is virtually alone in defining the MRS as the slope of the indifference curve rather than the absolute value of that slope. We adopt the latter convention because is by far the most prevalent.

transformation (strictly increasing function) of $U(\cdot,\cdot)$ represents her preferences as well [Varian (2006, Ch. 4)].

With all the pertinent concepts in place, we can now spell out the general properties that inertemporal preferences are assumed to satisfy:

1. An agent's preference scale has a utility representation. This ensures that that her preferences are *complete* and *transitive*, meaning that the agent can rank all conceivable consumption streams in a consistent fashion.

2. There is a utility function (representing agent's preferences) that is continuous and sufficiently differentiable with positive marginal utilities at any positive consumption stream. This premise ensures that preferences exhibit *strong monotonicity*—more is better[3].

3. A utility function as in Property 2 exhibits *diminishing marginal rate of substitution*—preferences and indifference curves (passing through positive consumption paths) are *strictly convex*. The strict convexity property conveys the premise that agents have a penchant for *consumption smoothing* over time or, equivalently, an aversion for consumption streams that vary over time.

It is worth remembering that as a consequence of Property 1 indifference curves **do not intersect** each other. By Property 2, indifference curves are **continuous** (have no breaks), **smooth** (have no kinks), and **slope downward**; further, indifference curves lying **farther** from the origin correspond to **higher** levels of utility. As already indicated, indifference curves are **strictly convex** by Property 3.

The notion of 'time preference' or 'impatience to consume' plays a central role in the Fisherian theory of interest. More precisely, an increase (decrease) in **impatience** for consumption now versus consuming later is signified by an increase (decrease) in the marginal rate of substitution, $MRS_{01}(c_0,c_1)$, **at any** positive consumption stream $(c_0,c_1)$—indifference curves get steeper (flatter) with greater (lower) impatience. It is customary in intertemporal analysis to adopt a time-additive form of lifetime utility with a fixed preference parameter $\delta$ that explicitly captures the agent's impatience, and is called the *subjective discount factor*[4]:

$$U(c_0,c_1) := u(c_0) + \delta u(c_1), \qquad 0 < \delta < 1. \tag{1.2}$$

It is assumed that the *period utility function* $u(\cdot)$ is continuous and twice-differentiable, and exhibits positive and diminishing marginal utility for all positive levels of consumption: $u'(c) > 0$ and $u''(c) < 0$ for all $c > 0$. It is easy to confirm that these assumptions about the period utility function ensure that the associated intertemporal utility function (1.2) is continuous and twice-differentiable, and satisfies Properties 2 and 3. Clearly, the parameter $\delta$ indicates how the

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[3] The continuity of the utility function also ensures the *continuity* of preferences—see, for example, Varian (1992, p. 95) for the elucidation of this concept.

[4] Sometimes one writes $\delta$ in the form $\delta = 1/(1+\rho)$, and the parameter $\rho := (1-\delta)/\delta$ is called the *rate of time preference*.

agent discounts utility from future consumption—a decline (increase) in the subjective discount factor would express an increase (decrease) in impatience.

### 1.2. Budget Constraints

As with the static model of consumer choice, the optimal consumption plan of the agent requires the knowledge of her market opportunities or budget constraints. The derivation of the latter along with the spelling out of the market structure of the model is taken up next.

Individuals can borrow or lend funds by issuing or purchasing one-period discount bonds, respectively, at date 0. The market for bonds is perfectly competitive. A bond here is a document that is issued at date 0 and confers upon its owner the right to one unit of account (for example, one dollar) one period from now, i.e. at date 1. This defines the unit of the bond; we presume that bonds are divisible, so that the number of units of bonds can take any real value. For instance, one can purchase 2.753 bonds. The future payment of one unit of account is the obligation of the issuer in exchange for having received the amount of $q$ dollars at date 0—the current unit price of the bond. [So it would cost $2.753q$ units of account to purchase 2.753 bonds.] Purchasing bonds amounts to making a **financial investment** at date 0 with a *one-period rate of return*

$$R := (1-q)/q; \tag{1.3}$$

this will be referred to as the *nominal interest rate*. We may anticipate that in equilibrium $0 < q \leq 1$ so that $R \geq 0$. Obviously, there is a one-to-one correspondence between the nominal interest rate and the price of bonds; any one of them is equally good for describing the bond market—which is often portrayed via several alternative names such as 'credit market', 'market for loans', 'market for loanable funds', 'capital market', and so on. In these notes, we take the interest rate $R$ as the key variable; once its value is known, the price of bonds can be determined by solving (1.3) with respect to the bond price to obtain

$$q = 1/(1+R). \tag{1.4}$$

Here the price of a bond is by definition equal to the *present value* of one unit of account (one dollar) delivered at date 1. If in addition to bonds, individuals could borrow and lend funds independently at an interest rate $R$ (say, through opening bank accounts), then, as we shall see later in class, Equation (1.4) would hold as an equilibrium condition.

As a precondition for a perfectly competitive bond market, we assume that all individually issued bonds are indistinguishable—agents never default on their debts, so there is no risk associated with holding bonds. Further, we have assumed that bonds are traded directly in the credit market—there are no financial intermediaries. In sum, our model abstracts from credit risk and financial intermediation. These aspects of reality would have to be introduced at a lower level of abstraction—after the fundamental theory of interest is grasped.

If an agent **purchases** $z$ bonds, then she is in effect **lending** an amount of $b$ dollars given by the value of purchased bonds, namely

$$b := qz. \tag{1.5}$$

Alternatively, $b$ is the dollar value of the agent's *financial investment*. So if the agent purchases $z$ bonds, then both $z$ and $b$ are **positive** numbers. By convention, a **negative** $z$ would indicate that the agent is a **borrower**, with $-z$ being interpreted as the number of bonds issued (and sold) and $-b = q(-z)$ as the amount of **borrowed funds**. Thus we may view $z$ and $b$ as *net demands*—they represent demands or supplies proper depending on whether they are positive or negative, respectively.

At each date there is a perfectly competitive spot market for goods, with $p_t > 0$ [5]designating the spot price level at date $t$ ($t$ = 0,1). When agents make decisions at date 0, they know the prices $q$ and $p_0$, and are assumed to correctly anticipate $p_1$. The choice variables $c_0, c_1$, and $z$ of an agent with endowment $\boldsymbol{\omega} := (\omega_0, \omega_1)$ is subject to two budget constraints: at each date there must be a balance between the agent's sources of funds and her use of these funds[6]. At date 0 the agent sells her initial endowment $\omega_0$ on the current spot market for goods on which the spot price level is $p_0$; this gives her a date 0 income or source of funds $p_0\omega_0$. It is assumed that the agent does not carry any bonds from the past, so $p_0\omega_0$ is her only source of funds at time 0. Her uses of funds at date 0 are consumption expenditures, $p_0c_0$, and bond purchases, $qz$. Thus the budget constraint for date 0 reads

$$p_0\omega_0 = p_0c_0 + qz. \tag{1.6}$$

At date 1 the agent will sell her endowment $\omega_1$ at the anticipated spot price $p_1$ to obtain the income $p_1\omega_1$. The purchase of $z$ bonds at date 0 yields a second source of funds at date 1 in the amount of $z$ dollars. The funds from these two sources will be used for the purchase of consumption $c_1$ at the spot price $p_1$. Hence the consumption expenditure $p_1c_1$ must satisfy the following budget constraint for date 1:

$$p_1\omega_1 + z = p_1c_1. \tag{1.7}$$

It is presumed that the agent leaves no bequests (so it does not wish to carry any bonds over the second period) and hence $p_1c_1$ is her only use of funds at date 1. If $z$<0, then the term $qz$ becomes a source of funds for (1.6)—it can be transferred to the left-hand side of Equation (1.6) as $-qz$, which is the amount of borrowed funds; at the same time the term $z$ becomes a use of funds for (1.7)—it can be transferred to the right-hand side of (1.7) as $-z$, which represents (as we shall see shortly) principal plus interest on borrowed funds. There is no money in the model (in the sense of store of value and medium of exchange) so there are no money holdings involved

[5] With strongly monotonic preferences, there cannot be free goods in equilibrium, so we may presume positive prices of goods when equilibrium prevails.

[6] More generally, we should require that the sum total of uses should not exceed the sum total of sources of funds. Since more is better in the preference scale of an agent, no funds will be left unused at her optimum.

in (1.6) and (1.7). It should be noted that the $z$ in (1.7) is $\$z = z \times \$1$, with \$1 being the face value of one unit of the bond.

Since we intend to describe the bond market in terms of the nominal interest rate rather the bond price, it useful to recast the constraints (1.6) and (1.7) in terms of the dollar amount $b$ of bonds purchased (or issued) at date 0, defined by Equation (1.5), and the nominal rate of interest $R$, linked to the price of bonds by (1.3) or (1.4). From (1.4) and (1.5) we have

$$z = b(1+R). \tag{1.8}$$

Insert (1.5) and (1.6) and (1.8) into (1.7) to obtain

$$p_0\omega_0 = p_0c_0 + b, \tag{1.9}$$

$$p_1\omega_1 + b(1+R) = p_1c_1. \tag{1.10}$$

Equations (1.5) and (1.8) make it clear that if the nominal interest rate is positive—that is, $R > 0$—then a given dollar amount of today's bonds translates into a larger dollar sum at date 1. Accordingly, agents regard a dollar's worth of income or expenditure differently depending on the date it arises—a phenomenon known as the **time value of money**. Thus \$1 received or spent now is equivalent to \$(1+$R$), its *future value,* at date 1. Conversely, \$1 received or spent in the future (i.e. at date 1) must be discounted by the one-period *discount factor*

$$d_1 := 1/(1+R) \tag{1.11}$$

to become comparable to dollars in the present[7]. When we discount by this factor—that is, when we multiply by $d_1$—we determine the *present value* of next date's dollar income or expenditure. Thus the *present value* (PV) of a cash flow stream $(CF_0, CF_1)$ is given by

$$PV = CF_0 + d_1CF_1, \tag{1.12}$$

where $CF_t$ is the cash flow (in or out of the agent) at date $t$, $t = 0,1$, and $d_1$ is given by (1.11).

Note that the discount factor is dimensionless. Even though it is numerically equal to $d_1$, the present value of \$1 received or spent at date 1 is equal to $d_1 \times \$1$ or $d_1$ dollars and hence is dimensionally distinct. Thus we have $q = d_1 \times \$1$.

Purchasing bonds is thus far the only opportunity for individual investment; hence the value $b$ of purchased bonds coincides with the *current nominal saving* $p_0\omega_0 - p_0c_0$ of the agent.

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[7] Following Luenberger (1998), the discount factors associated with each date $t$ are denoted by $d_t$. These are the factors by which future cash flows must be multiplied to obtain an equivalent present value at date 0.

By solving Equation (1.10) with respect to $b$ we obtain

$$b = (p_1 c_1 - p_1 \omega_1)/(1+R). \tag{1.13}$$

Thus the budget constraint for date 1 in effect dictates that current nominal saving must be equal to the present value of the excess future consumption spending over future income. Or, equivalently, that this excess of future spending over future income must equal the future value of the agent's current saving. Should future consumption spending be less than future income, it must equal the future value of current borrowing.

Insert the expression (1.13) for $b$ into (1.9) and rearrange terms to obtain the agent's *intertemporal* or *lifetime budget constraint* (in nominal terms):

$$p_0 c_0 + \frac{p_1 c_1}{1+R} = p_0 \omega_0 + \frac{p_1 \omega_1}{1+R}. \tag{1.14}$$

The graph of (1.14) is a straight line in the $c_0 c_1$ plane, known as the *intertemporal budget line*. The intertemporal budget constraint requires that the present value of the agent's consumption expenditures over time be equal to the present value of her income stream (obtained from the sale of her time-distributed endowment in the spot markets for goods at dates 0 and 1). It should be clear that we may represent the constraints facing the agent in two alternative ways: either by the system of equations (1.9) and (1.10) or the equivalent system of equations (1.9) and (1.14). The slope $dc_1 / dc_0$ of the intertemporal budget line is the opportunity cost of an increment of current consumption in terms future consumption forgone; it can be calculated by solving (1.14) for $c_1$ in terms $c_0$ and taking the derivative of $c_1$ with respect to $c_0$:

$$\frac{dc_1}{dc_0} = -\frac{p_0}{p_1}(1+R). \tag{1.15}$$

## 1.3. Nominal and Real Interest Rates

To interpret the right-hand side of (1.15) we need to introduce the real rate of return on bonds, i.e. the rate of return adjusted for inflation. To this end, notice that the purchase of a bond at date 0 amounts to foregoing $q / p_0$ units of current consumption in exchange for $1/ p_1$ units of consumption at date 1, so a real rate of return is involved. The *real interest rate, r* is defined as the *real rate of return from investment in bonds*:

$$r := \frac{\frac{1}{p_1} - \frac{q}{p_0}}{\frac{q}{p_0}}. \tag{1.16}$$

In effect, equation (1.16) defines the *expected* or *actual* real interest rate depending on whether the price $p_1$ signifies the expected (at date 0) or actual spot price of corn at date 1, respectively. Since, by assumption, our agents at date 0 anticipate correctly the spot price of corn at date 1, the expected or *ex ante* real interest rate coincides with the actual or *ex post* real interest rate. We may rewrite (1.16) as $1+r=(p_0/p_1)/q$ which, in view of (1.4), yields

$$1+r=(1+R)(p_0/p_1). \tag{1.17}$$

Recalling that $R \geq 0$ and $p_0/p_1 > 0$, it follows from (1.17) that $1+r>0$. Thus the real interest rate can be negative but must stay above -1. In view of (1.17), (1.15) becomes

$$\frac{dc_1}{dc_0}=-(1+r). \tag{1.18}$$

Accordingly, it is the real interest rate, rather than the nominal interest rate, that determines the increment in consumption at date 1 that an agent can get if she foregoes a unit of consumption at date 0. To put it otherwise, it is the real interest rate that counts in the consumption/saving decision of the agents.

That the spot price $p_1$ is correctly anticipated at time 0 is tantamount to correctly forecasting the *inflation rate,* denoted by the Greek letter $\iota$ (iota):

$$\iota := \frac{p_1-p_0}{p_0}. \tag{1.19}$$

Hence $1+\iota = p_1/p_0$ and (1.17) reads

$$1+r=\frac{1+R}{1+\iota}, \tag{1.20}$$

which is known as the *Fisher identity*[8]. It is often useful to rewrite (1.20) in an approximate form. To this end, multiply both sides of (1.20) by $(1+\iota)$ to obtain

$$1+R=(1+r)(1+\iota)=1+r+\iota+\iota r;$$

if $\iota$ and $r$ are sufficiently small fractions, then $\iota r$ is negligible and hence

$$r \cong R-\iota. \tag{1.21}$$

[8] The Fisher hypothesis is the suggestion that a change in the inflation rate has no effect on the real interest rate. This hypothesis is false in the Keynesian short run, but does hold true in the medium run—see Blanchard (2006).

That is, the real rate of interest is approximately equal to the nominal interest rate minus the rate of inflation[9].

The Fisherian theory of interest is concerned with the determination of the real rate of interest. The premise here—a version of the classical dichotomy[10]—is that the real rate of interest is determined by the real side of the economy (entailing the Fisher hypothesis reviewed in Footnote 8), while the inflation rate is determined by the monetary side, more specifically, by the growth rate of the money supply[11]. This dichotomy is not expected to hold in the Keynesian short-run—changes in the stock of money do have real effects. The dichotomy holds in the medium-run as defined, for example, by Blanchard (2006).

### 1.4. Budget Constraints in Real Terms

Given that the real interest rate is all that our model is expected to determine, we may recast the budget constraints in real terms and bring in directly the real interest rate. We begin by rewriting the one-date budget constraints (1.9) and (1.10) in real terms. Dividing both sides of (1.9) by $p_0$ (which is assumed to be positive) and solving for $c_0$ we have

$$c_0 = \omega_0 - \frac{b}{p_0}. \tag{1.22}$$

In a similar fashion, divide both sides of (1.10) by $p_1$, take into account (1.17), and solve for $c_1$ to obtain

$$c_1 = \omega_1 + (1+r)\frac{b}{p_0}. \tag{1.23}$$

As expected, it is the real value $b/p_0$ rather than the nominal value $b$ of purchased or issued bonds that determines the consumption path $(c_0, c_1)$. Instead of introducing a new symbol for $b/p_0$ we may take date 0 corn as the *numeraire* by setting $p_0 = 1$. Accordingly, (1.22) and (1.23) read

$$c_0 = \omega_0 - b, \tag{1.24}$$

$$c_1 = \omega_1 + (1+r)b. \tag{1.25}$$

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[9] The approximation becomes better the shorter the length of the period; and (1.19) becomes an exact relation as the length of the period approaches zero.
[10] See, for example, Williamson (2005, Ch. 10).
[11] More precisely, this rate must be adjusted by the rate of growth of the economy's output.

Here and in the sequel $b$ stands for the volume of financial investment (or borrowing) expressed in units of goods. Finally, divide both sides of equation (1.14) by $p_0$ and take into account (1.17) to obtain the intertemporal budget constraint restated in real terms:

$$c_0 + \frac{c_1}{1+r} = \omega_0 + \frac{\omega_1}{1+r}. \tag{1.26}$$

We would also arrive at (1.26) by solving (1.25) for $b$ and inserting the resulting expression into (1.24), as expected. The right-hand side of (1.26) is defined as the agent's *endowed real wealth* $W_0$ at date 0:

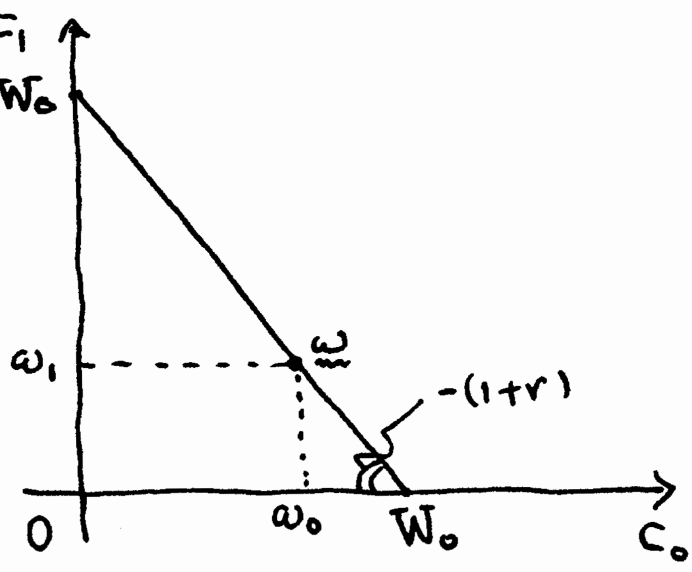
$$W_0 := \omega_0 + \frac{\omega_1}{1+r}. \tag{1.27}$$

Thus we may rewrite (1.26) in the form

$$c_0 + \frac{c_1}{1+r} = W_0. \tag{1.28}$$

The graph of (1.28) is portrayed in Fig. 1.1.

Fig. 1.1 The Intrtemporal Budget Line.




It is clear from (1.28) and its graph in Fig. 1.1 that $W_0$ is the maximum attainable current consumption if the agent consumes nothing in the second period $(c_1 = 0)$; it exceeds her current endowed income by the amount she can borrow against future endowed income[12]. The future

[12] The time-distributed endowment consists of quantities of corn or 'consumption claims'. From the sale of her endowed amounts of corn in the spot markets for goods the agent obtains the nominal income stream

value $(1+r)W_0$ of the agent's endowed wealth is the maximum attainable future consumption if she consumes nothing during the first period $(c_0 = 0)$; it exceeds her future endowed income by accrued principal plus interest from the spending of her current endowment in purchasing bonds.

Two remarks are in order. First, by (1.24) the amount of financial investment coincides with the agent's current *real saving s*,

$$s := \omega_0 - c_0 = b. \tag{1.29}$$

Second, equations (1.24) and (1.25) constitute a parametric representation of the intertemporal budget line (1.26). The parameter here is *b*; its range is restricted by the logical requirement that consumption at any date cannot be negative. By (1.24), the requirement $c_0 \geq 0$ implies that $b \leq \omega_0$; while from (1.25) and $c_1 \geq 0$ we infer that $b \geq -\omega_1/(1+r)$. As *b* varies from $b = \omega_0$ down to $b = -\omega_1/(1+r)$, the plan $(c_0, c_1)$ specified by (1.24) and (1.25) traces out the intertemporal budget line in Fig. 1.1.

Thus far the inflation rate is a predetermined constant. We may presume from now on (without loss of generality) that $\imath = 0$ and hence, by (1.20), $r = R$; as a consequence, (1.4) becomes

$$q = \frac{1}{1+r}, \tag{1.30}$$

where $q$ is now expressed in units of goods (per one unit of the bond); further, from (1.17) we obtain $p_1 = p_0 = 1$ (on account of our choice of corn at date 0 as the numeraire). In other words, by purchasing one unit of the bond at date 0 one foregoes one q units of consumption now in exchange of receiving one unit of consumption at date 1. In view of (1.30), (1.28) looks like an agent's static budget line in a two good world with given endowment of goods—the buying and selling model of consumer choice [Varian (2006), Ch. 9)]. The two goods here are corn available at date 0 and contracts (bonds) for corn at date 1; the price of the former is normalized to 1, while the price of the latter is $q$.

Before concluding this section, it is worth noting that corresponding to the present value (1.12) of a cash flow stream $(CF_0, CF_1)$ there is the real present value counterpart

$$PV / p_0 = \frac{CF_0}{p_0} + \hat{d}_1 \frac{CF_1}{p_1}, \tag{1.31}$$

where

---

$(p_0\omega_0, p_1\omega_1)$; the associated real income stream $(\omega_0, \omega_1)$ coincides with the time-distributed endowment of goods.

$$\hat{d}_1 := 1/(1+r) \quad (1.32)$$

is now the *real discount factor* that coverts amounts of real spending or real income at date 1 to date 0 equivalent payments in consumption units; $PV/p_0$ is the real present value of the sequence $(CF_0/p_0, CF_1/p_1)$ of real payments ( or receipts )[13]. These two ways of writing present values, (1.12) and (1.31) are equivalent. We may compute the present value as the nominal or dollar present value of a cash flow stream expressed in dollars, discounting using the nominal interest rate; or as the real present value of the cash flow stream expressed in real terms, discounting using the real interest rate. It is the second way that we shall naturally adopt in the subsequent sections. All sequences of spending and incomes will be expressed in real terms; only the real interest rate and real present or future values will figure in the discussion. Accordingly, the adjective 'real' will often be dropped from now on without danger of confusion.

## 2. The Optimal Consumption Plan

An agent's problem of choice is to select the values of $b, c_0,$ and $c_1$ that maximize lifetime utility $U(c_0, c_1)$ subject to the one-date budget constrains (1.24) and (1.25). Observe, though, that the latter equations fully specify the consumption plan of the agent once the amount of financial investment (or borrowing) is selected—the choice of the triple $(b, c_0, c_1)$ boils down to the selection of the volume of financial investment. Substituting the expressions (1.24) and (1.25) for $c_0$ and $c_1$ into the utility function $U = U(c_0, c_1)$ we make $U = U(c_0, c_1)$ a **composite function** of $b$ and hence the agent's constrained-maximization problem over $(b, c_0, c_1)$ reduces to the **unconstrained problem** of maximizing $U(c_0, c_1)$ over $b$. Assuming an interior maximum[14], i.e. assuming that the maximizing value of $b$ lies in the interval $-\omega_1/(1+r) < b < \omega_0$, the **first-order condition** for maximizing $U(c_0, c_1)$ over $b$ reads[15]

[13] Expressions (1.31) and (1.32) are derived for arbitrary given inflation rate and prices $p_0$ and $p_1$ measured in arbitrary units of account.

[14] This amounts to assuming that date 0 consumption and date 1 consumption are both positive at the optimum. By inspection of Fig. 2.1, we may ensure an interior maximum by positing that any indifference curve passing through a positive consumption stream does not intersect the coordinate axes. This additional restriction on preferences will be taken from granted from now on.

[15] See Sydsaeter and Hammond (2006), Section 8.1. Suppose $v$ is a function of $x$ and $y$, with $v = F(x, y)$, where $x$ and $y$ both are functions of a variable t, with $x = f(t)$, $y = g(t)$, then $v$ is a **composite function** of $t$, $v = F(f(t), g(t))$; its **total derivative** is given by the **chain rule**:

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$ [Sydsaeter and Hammond (2006), Section 12.1].

$$\frac{dU}{db} = \frac{\partial U}{\partial c_0}\frac{dc_0}{db} + \frac{\partial U}{\partial c_1}\frac{dc_1}{db} = 0. \tag{2.1}$$

From (1.24) and (1.25) we derive $dc_0 / db = -1$ and $dc_1 / db = 1 + r$; hence (2.1) yields

$$\frac{\partial U}{\partial c_0} = \frac{\partial U}{\partial c_1}(1 + r). \tag{2.2}$$

Equation (2.2) expresses the standard marginal condition for an optimum: $\frac{\partial U}{\partial c_0}$ is the loss in utility if the agent lends (and therefore foregoes the consumption of) another unit of corn at date 0; $\frac{\partial U}{\partial c_1}(1 + r)$ is the increase in utility she obtains from the extra $1 + r$ units of corn at date 1[16]. The agent continues to lend or borrow (buy or sell bonds) until the marginal loss equals the marginal gain[17].

To interpret the first-order condition geometrically, divide both sides of (2.2) by $\partial U / \partial c_1$ [which, recall, is assumed to be positive at any positive consumption stream] and recall (1.1) to infer that (2.2) is equivalent to the familial **tangency condition**

$$MRS_{01}(c_0, c_1) = 1 + r. \tag{2.3}$$

The optimal consumption plan is portrayed in Fig. 2.1.

Equation (2.3) is a restatement of the marginal condition for an optimum: $MRS_{01}(c_0, c_1)$ is the loss in future consumption if the agent lends (and therefore foregoes the consumption of) another unit of corn at date 0; $1 + r$ is the increase in consumption she obtains at date 1. Once again, the agent continues to lend or borrow (buy or sell bonds) until the marginal loss equals the marginal gain. We may also interpret equation (2.3) as a requirement of balance between the agent's

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16 One lends one unit of consumption at date 0 by purchasing $1/q$ units of the bond; the associated payoff is $1/q = 1 + r$ units of consumption at date 1.

17 In view of (1.30), the first-order condition (2.2) may also be written in the form

$$q\frac{\partial U}{\partial c_0} = \frac{\partial U}{\partial c_1}.$$

Here $q\frac{\partial U}{\partial c_0}$ is the marginal loss in utility if the agent buys another unit of the bond; $\frac{\partial U}{\partial c_1}$ is the increase utility she obtains from the payoff of the bond (i.e. one unit of consumption) at date 1. The agent continues to buy or sell bonds until the marginal loss equals the marginal gain.

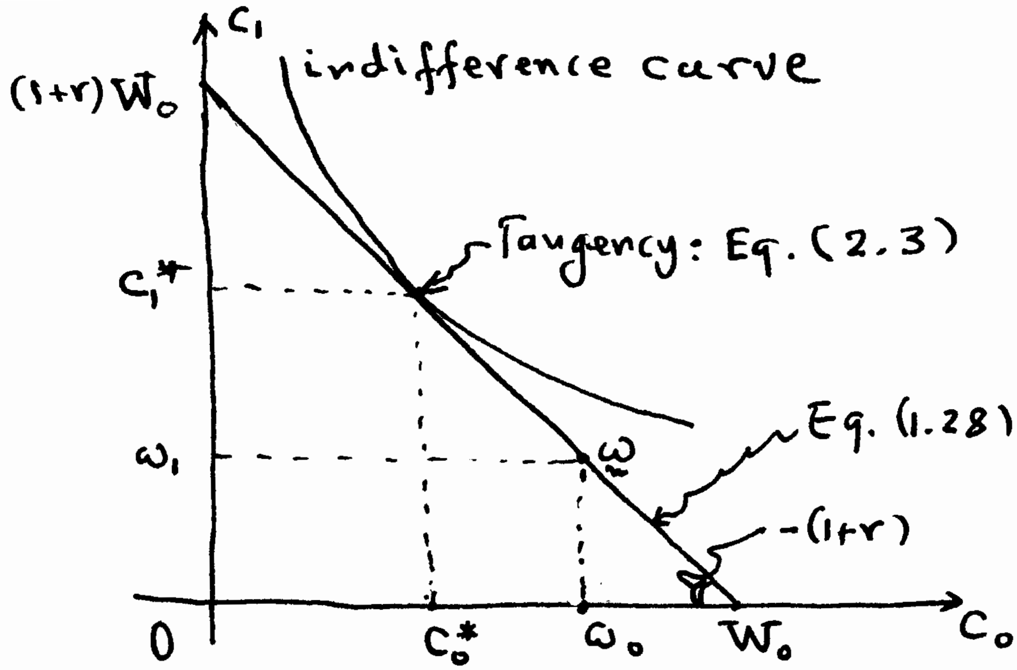


Fig. 2.1 The Optimal Consumption Plan

marginal willingness to pay for a one-unit increase in current consumption, $MRS_{01}(c_0, c_1)$, and the opportunity cost $1+r$ of that increase (both expressed in terms of future consumption).

From Fig. 2.1 we infer that the premise of diminishing $MRS_{01}$ or strictly convex indifference curves ensures that the first-order condition (2.2) yields indeed a maximum, and that the utility-maximizing consumption plan is uniquely determined for any given $r > -1$. In reference to the agent's utility-maximizing problem we have **three endogenous variables**: $b, c_0$, and $c_1$. The optimal values of these variables are in principle determined by the system of **three** equations: the one-period budget constraints (1.24) and (1.25) and the tangency condition (2.3). From our discussion in the preceding section, this system is equivalent to the system of equations (1.24), (1.26), and (2.3). The last two equations determine the optimal values of $c_0$ and $c_1$ as functions of the real interest rate *r*, the agent's *demand functions for present and future consumption*[18]:

$$c_0 = c_0^*(r) \text{ and } c_1 = c_1^*(r) \qquad (r > -1). \tag{2.4}$$

Inserting $c_0 = c_0^*(r)$ into (1.24) we obtain the optimal value of *b* as a function of the real interest rate *r*, the agent's *net demand function for bonds or financial investment*:

$$b = b^*(r) = \omega_0 \text{-} c_0^*(r) \qquad (r > -1). \tag{2.5}$$

The graph of the net demand function (2.5) is called the *net demand curve for bonds*. The arguments in the functions (2.4) and (2.5) should include the endowed real incomes $\omega_0$ and $\omega_1$ of

[18] These demand functions are positive-valued on account of Footnote 15.

the agent. It is customary, though, to suppress the dependence on $\omega_0$ and $\omega_1$ to underscore the fact that $\omega_0$ and $\omega_1$ play the role of 'shift parameters'—a change in their values shifts the graphs of the functions (2.4) and (2.5)[19]. In light of the notation adopted in (2.4) and (2.5), the designation of an optimal solution by $(b^*, c_0^*, c_1^*)$ is an abbreviation for the optimal solution $(b^*(r), c_0^*(r), c_1^*(r))$ associated with an arbitrary given interest rate $r$.

The agent's *real saving* is defined as the excess of her current endowed output over her current consumption: $s := \omega_0 - c_0$. Accordingly, the agent's *real saving function* is defined by

$$s = s^*(r) := \omega_0 \text{-} c_0^*(r) \qquad (r > -1); \tag{2.6}$$

and its graph is referred as the *saving curve*. Since the purchase of bonds is the only investment opportunity in a pure exchange environment, the agent's saving function coincides with her net demand function for bonds. Indeed, from (2.5) and (2.6) have[20]

$$b^*(r) \equiv s^*(r). \tag{2.7}$$

In order to solve the system of (1.24), (1.26), and (2.3), it is advisable to proceed in two steps. In the first step, solve the system of equations (1.28) and (2.3) for $c_0$ and $c_1$ in terms of $1+r$ and $W_0$:

$$c_0 = \bar{c}_0(1+r, W_0) \text{ and } c_1 = \bar{c}_1(1+r, W_0), \tag{2.8}$$

where we have used overbars to designate the solution values. In turn, substitute $W_0$ from (1.27) in the expressions for $c_0$ and $c_1$ to obtain (2.4). That is,

$$c_0^*(r) = \bar{c}_0(1+r, \omega_0 + \frac{\omega_1}{1+r}) \text{ and } c_1^* = \bar{c}_1(1+r, \omega_0 + \frac{\omega_1}{1+r}). \tag{2.9}$$

The writing of the consumption demand functions in the form (2.8) is theoretically important in its own right. For it allows to examine the impact of a change in wealth while $r$ is held constant on the optimal solution and the level of maximum utility. The concept of a normal good is defined in terms of (2.8): consumption at date $t$ is said to be a *normal good* if $\partial c_t / \partial W_0 > 0$ for

[19] We may, of course, include the parameters $\omega_0$ and $\omega_0$ in the arguments of the functions (2.4) and (2.5) via the standard notation $c_t^*(r; \omega_0, \omega_1)$ and $b^*(r; \omega_0, \omega_1)$. The semicolon signifies the separation of variables and parameters in the arguments of a function—the symbols before the semicolon denote variables, while those after it denote parameters.

[20] Recall the symbol := means 'by definition equal', while the three-line equality sign ≡ stands for 'identically equal'. So (2.7) means that $b^*(r) = s^*(r)$ **for all** $r > -1$.

all $r > -1$ and $W_0 > 0$. Insert (2.8) into (1.28) and partially differentiate with respect to $W_0$ to obtain

$$\frac{\partial c_0}{\partial W_0} + \frac{1}{1+r}\frac{\partial c_1}{\partial W_0} = 1. \tag{2.10}$$

Clearly, for this identity to hold true at least one of the partial derivatives in it must positive and hence consumption at some date (at least) must behave like a normal good.

A second significant result pertains to the variation of maximum utility with a change in wealth (holding the real interest rate constant). Insert (2.8) into $U = U(c_0, c_1)$ to obtain the maximum value $\overline{U}$ of the utility function as a composite function of $1+r$ and $W_0$[21]:

$$\overline{U} = \overline{U}(1+r, W_0) := U(\overline{c}_0(1+r, W_0), \overline{c}_1(1+r, W_0)). \tag{2.11}$$

Partially differentiate (2.11) with respect to $W_0$ to derive[22]

$$\frac{\partial \overline{U}}{\partial W_0} = \frac{\partial U}{\partial c_0}\frac{\partial c_0}{\partial W_0} + \frac{\partial U}{\partial c_1}\frac{\partial c_1}{\partial W_0}; \tag{2.12}$$

in turn, solve the first-order condition (2.2) for $\partial U / \partial c_1$, insert the resulting expression into (2.12), and take into account (2.10) to obtain

$$\frac{\partial \overline{U}}{\partial W_0} = \frac{\partial U}{\partial c_0}, \tag{2.13}$$

with the understanding that the marginal utility on the right is evaluated at the optimal values given in (2.8). Since the optimal consumption stream has been assured to be positive and marginal utilities are assumed to be positive at all positive consumption plans (so as to ensure strong monotonicity) we infer that

$$\frac{\partial \overline{U}}{\partial W_0} > 0 \tag{2.14}$$

for all $r > -1$ and $W_0 > 0$. This is an extremely important conclusion: maximum utility increases with greater wealth (holding the real interest rate constant)—**more wealth is better**. Do not confuse this fundamental result with the mere assumption of strong monotonicity—more

[21] Known as the **indirect utility function.**

[22] You need to invoke the pertinent chain rule of differentiation; see Sydsaeter and Hammond (2006), Section 12.2. The partial derivative $\partial \overline{U} / \partial W_0$ is called the 'marginal utility of wealth'—'marginal indirect utility of wealth' would be a better name.

consumption at date 0 or at date 1 is better. To be able to infer that more wealth is better, you need **all** assumptions about intertemporal preferences, a perfectly competitive market environment, and the premise of utility-maximizing behavior. While this should be clear from the sequence of arguments that have led to (2.14), you may reinforce such an understanding by providing a careful geometrical demonstration of (2.14). Begin with Fig. 2.1 and consider a small increment in wealth, $\Delta W_0 > 0$. Draw the intertemporal budget line corresponding to the wealth $W_0 + \Delta W_0$, and draw the associated new point of tangency. Infer that the maximum utility level associated with $W_0 + \Delta W_0$ is greater than the original on the grounds that the agent's utility function exhibits positive marginal utilities and diminishing $MRS_{01}(c_0, c_1)$ [strictly convex indifference curves].

When we wish to examine the impact of a change in the interest rate on the optimal solution, we **cannot** hold wealth constant—it has to change according to (1.27). The optimal values of $(b, c_0, c_1)$ must be expressed in terms of the interest rate alone to obtain the functions (2.4)-(2.6) [via (2.9)]. To get a sense of how the functions (2.4)-(2.6) and (2.8) look like, consider the special case where the agent's utility function is of the Cobb-Douglas form:

$$U(c_0, c_1) := c_0^{\alpha} c_1^{\beta} \qquad (\alpha, \beta > 0). \tag{2.15}$$

With this type of utility function, the system of (1.28) and (2.3) [along with he definitional equation (1.27)] yields[23]

$$c_0 = \frac{\alpha}{\alpha + \beta} W_0 = \frac{\alpha}{\alpha + \beta}\left(\omega_0 + \frac{\omega_1}{1 + r}\right), \tag{2.16}$$

$$c_1 = \frac{\beta}{\alpha + \beta} W_0 (1 + r) = \frac{\beta}{\alpha + \beta}\left[\omega_0 (1 + r) + \omega_1\right]. \tag{2.17}$$

Hence from (2.5) and (2.6) we derive

$$b^*(r) \equiv s^*(r) = \frac{1}{\alpha + \beta}\left(\beta \omega_0 - \frac{\alpha \omega_1}{1 + r}\right). \tag{2.18}$$

We shall restrict our attention to the case of a positive endowed income stream[24], namely $\omega_0 > 0$ and $\omega_1 > 0$. It is then easy to show that the saving function (2.18) is increasing and strictly concave [$ds^* / dr > 0$ and $d^2 s^* / dr^2 < 0$] for all $1 + r > 0$ or $r > -1$. Further, observe that $b^*(r) \equiv s^*(r) \to \frac{\beta \omega_0}{\alpha + \beta}$ as $r \to +\infty$ and that $b^*(r) \equiv s^*(r) \to -\infty$ as $r \to -1$. These properties of

[23] The derivation of (2.16) and (2.17) is left as an exercise.

[24] This should be presumed to be the case for the 'representative agent'—see Section 7. The study and interpretation of (2.18) for the cases where $\omega_0 > 0$ and $\omega_1 = 0$ or $\omega_0 = 0$ and $\omega_1 > 0$ is left as an exercise.

the saving function are reflected by its graphs in Fig. 2.2, drawn for three different ranges of parametric values.

To fully appreciate the significance of these alternative ranges, it is useful to introduce at this point the concept of the *autarky* (or *no-borrowing, no-lending*) *interest rate* $\bar{r}$ of the agent, which is defined by the condition

$$MRS_{01}(\omega_0, \omega_1) = 1 + \bar{r} . \qquad (2.19)$$

In other words, when $r = \bar{r}$, the tangency point in Fig. 2.1 is the agent's endowment—the autarky rate $\bar{r}$ is that interest rate at which the optimal consumption plan of the agent is her endowed income stream. Accordingly, it also satisfies (and can be calculated from any of) the following conditions:

$$c_0^*(\bar{r}) = \omega_0 , \quad b^*(\bar{r}) = s^*(\bar{r}) = 0 . \qquad (2.20)$$

If both $\omega_0$ and $\omega_1$ are positive, then the left-hand side of (2.19) is a positive number; hence the autarky interest rate is a well defined number $\bar{r} > -1$. When the utility function is of the form (2.15), we can calculate $\bar{r}$ by setting the expression in (2.18) equal to zero. We have

$$\bar{r} = \frac{\alpha\omega_1}{\beta\omega_0} - 1 . \qquad (2.21)$$

Thus the magnitude of $\bar{r}$ (and the position of the saving curve, the graph of (2.18)) depends on the relation between $\alpha\omega_1$ and $\beta\omega_0$. See Fig. 2.2.

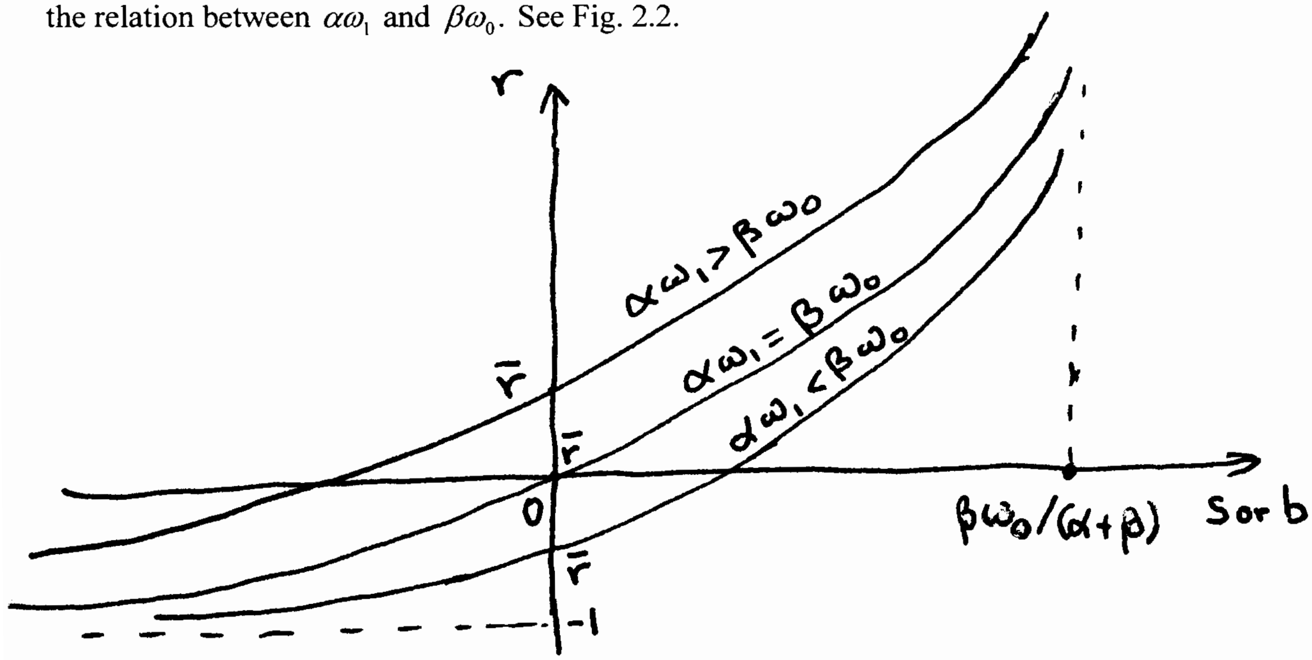


Fig. 2.2 Net Demand for Bonds and Saving Curves Associated with Cobb-Douglass Preferences

By inserting $\alpha\omega_1 = \beta\omega_0(1+\bar{r})$ [from (2.21)] into (2.18) we may rewrite the latter in the form

$$b^*(r) \equiv s^*(r) = \frac{\beta\omega_0}{\alpha+\beta}\left(1-\frac{1+\bar{r}}{1+r}\right). \tag{2.22}$$

Equation (2.22) shows that the agent chooses to be a borrower (or supplier of bonds) for $r < \bar{r}$ and a lender (or purchaser of bonds) for $r > \bar{r}$.

It turns out that the broad characteristics of the saving function associated with Cobb-Douglas preferences hold true more generally—that is, for all preferences with standard neoclassical properties as described in Section 1: the preferences of the agent are represented by a continuous and sufficiently differentiable utility function which exhibits positive marginal utilities and diminishing $MRS_{01}$ for all positive consumption streams[25]. In this general setting, the properties of the functions (2.4)-(2.6) are deciphered through an analysis of the substitution and wealth effects (on the optimal choice of the agent) of a change in the interest rate[26].

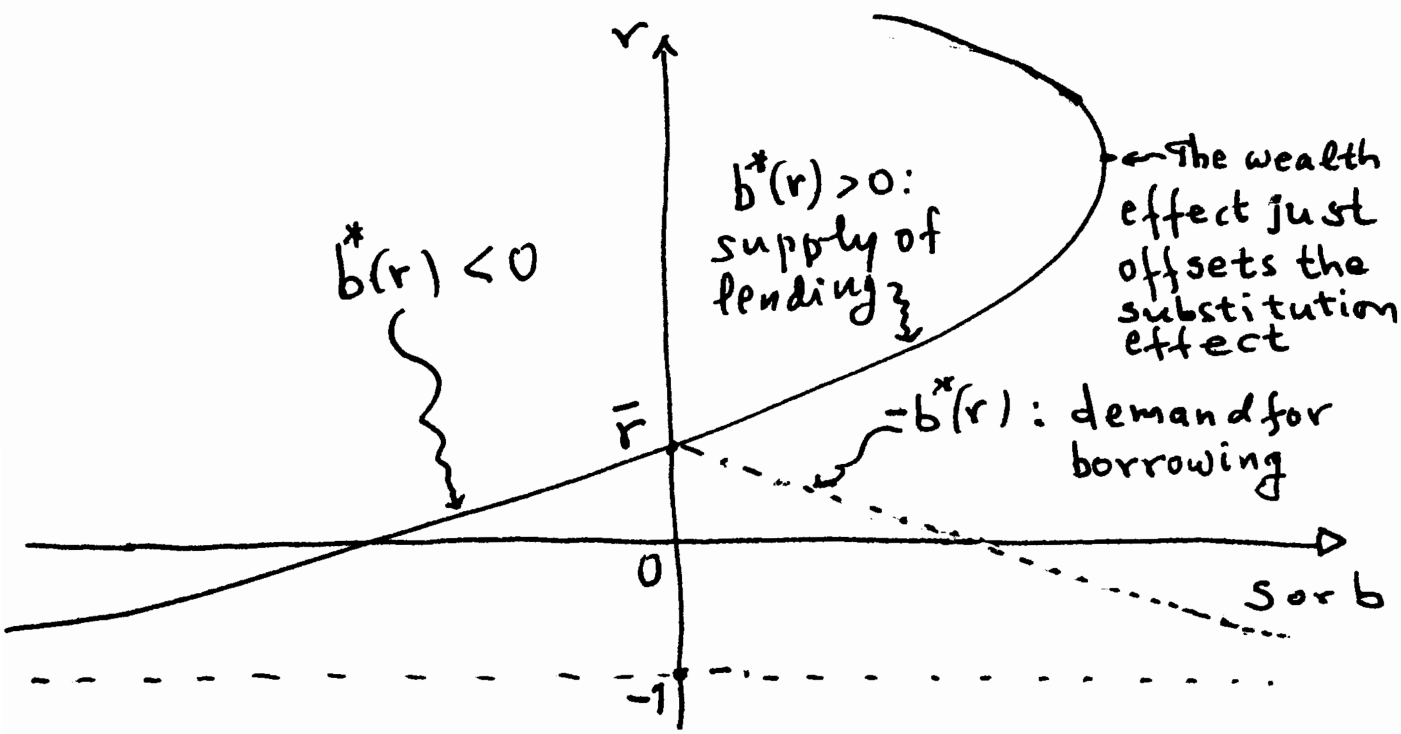


Fig. 2.3 Saving and Net Demand for Bonds

[25] We need to add the assumption that consumption at date 0 is a normal good. This is a mild restriction in view of the fact that at least one commodity must be a normal good.
[26] See Varian (2003, Ch. 10) and Hirshleifer and Hirshleifer (1998, Ch. 14) for the details of such an analysis.

The substitution effect of a change in the interest rate is unambiguous. For instance, an increase in the interest rate raises the relative cost of current consumption in terms of future consumption foregone and hence the agent would substitute away from current consumption—it reduces current consumption and hence raises the volume of saving and financial investment (or reduces the amount of borrowing). Since current consumption is assumed to be a normal good, the wealth effect of a higher interest rate reinforces the substitution effect if the agent happens to be originally a borrower; but acts in the opposite direction if the agent happens to be a lender (purchaser of bonds) at the original interest rate. While the substitution effect is expected to dominate the wealth effect for a significant range of interest rate values above the autarky level, the possibility of reversal and the onset a 'backward bending' saving curve at a sufficiently high interest rate cannot be precluded. See Fig. 2.3.

## 3. Production and Investment in Physical Assets

Suppose now that in addition to the opportunity of purchasing (or issuing) bonds at date 0, a typical agent has also the option of allocating a part of her current income $\omega_0$ to *physical* or *real investment*—to the purchase of corn from the goods market to be used as capital input . With $k \geq 0$ units of physical capital purchased at date 0, the agent can carry out production to obtain $y_1$ units of goods at date 1 according to an *intertemporal production function*

$$y_1 = f(k), \tag{3.1}$$

reflecting her technological know-how. Inputs other than physical capital are being ignored or kept frozen for the sake of simplicity. Further, physical capital is assumed to be 'circulating capital'—it gets completely used up during the production period (which extends from date 0 to date 1). This is why the capital in place and the amount of new capital goods (i.e. productive investment) purchased at date 0 coincide. Circulating capital is like seeds of corn with 100% depreciation over the first period. The production function is assumed to have the following properties[27]:

1. There is no output without the use of some input: $f(0) = 0$.

2. $f(\cdot)$ is continuous and twice-differentiable for all positive values of the capital input.

3. $f(\cdot)$ exhibits positive **marginal product of capital** for all positive values of the capital input. That is,

$$\frac{dy_1}{dk} = f'(k) > 0 \qquad \text{for all } k > 0. \tag{3.2}$$

[27] For the concept of the production function and the intuitive rationale for its properties, see Varian (2006), Ch. 18.

4. $f(\cdot)$ exhibits **diminishing marginal product of capital** for all positive values of the capital input. That is,

$$\frac{d^2 y_1}{dk^2} = f''(k) < 0 \quad \text{for all } k > 0. \tag{3.3}$$

We often refer to a production function that satisfies properties [1] to [4] as **neoclassical**[28]. The graph of a neoclassical production function is portrayed in Fig. 3.1.

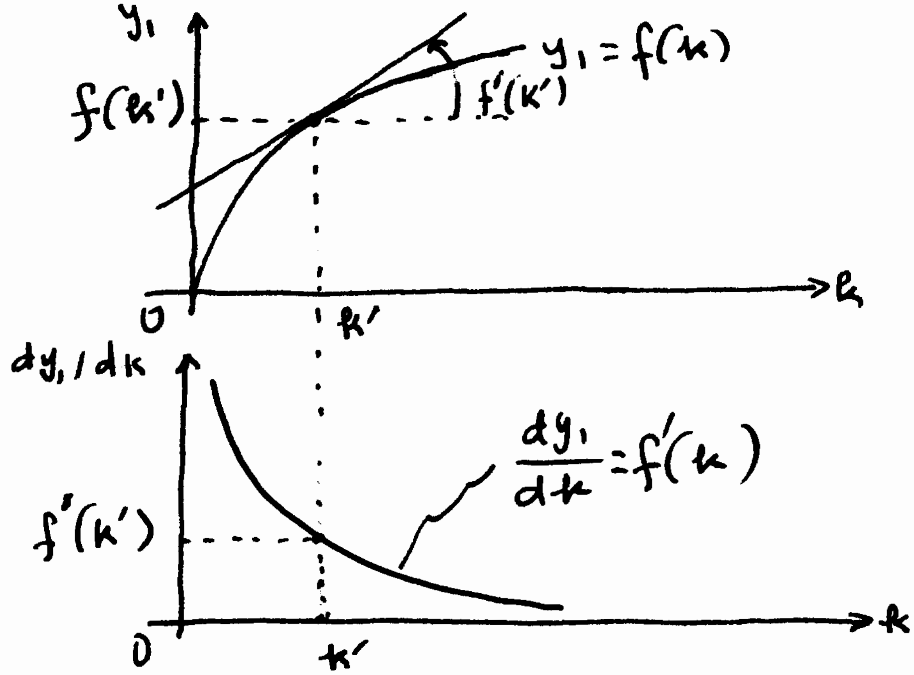


Fig. 3.1 The Neoclassical Production Function

To illustrate, $y_1 = \sqrt{k}$ and $y_1 = k^{\alpha}$ for $0 < \alpha < 1$ are examples of neoclassical production functions.

It should be noted that we have added the possibility of production to the model without introducing separate units (such as business firms) that specialize in production. Instead, each agent is to function as a consumer, trader, investor, and producer—she is endowed with technological know-how (her production function) and she is able to carry out production. Hence each agent is now characterized be three elements (treated as exogenous data):

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[28] A neoclassical production function is usually required to satisfy an additional requirement, known as the Inada conditions: the marginal product of capital $dy_1/dk$ 'blows up' as $k$ gets close to zero (i.e. $dy_1/dk \to +\infty$ as $k \to 0$) and gets very small as $k$ becomes very large (i.e. $dy_1/dk \to 0$ as $k \to +\infty$).

• Her preferences over alternative consumption plans $(c_0, c_1)$, represented by a utility function $U = U(c_0, c_1)$;

• Her endowed income stream $(\omega_0, \omega_1)$; and

• Her production function $f(\cdot)$.

The characteristics of each agent are always assumed to satisfy the properties we have posited regarding preferences, endowments, and production functions.

Our economy consists of a community of $N$ agents, each specified by the above three characteristics, along with the market structure introduced in the preceding sections. It is a two-date (or two-period) finance economy with production. That agents are posited to be consumers, traders, investors, and producers at the same time reflects a theoretical strategy rather an intention of direct description of actual economies. The objective is to provide a body of theory that will render useful insights about the operation of the latter. The theoretical approach that underlies our discourse on the Fisherian model is clearly articulated (among others) by Robert C. Merton (1982). The following passage (p. 601) summarizes this theoretical strategy and succinctly describes the objectives of investment theory as a field of financial economics:

> *Investment theory* is the study of the individual behavior of households and economic organizations in the allocation of their resources to the available investment opportunities. For the purposes of investment theory, economic organizations are characterized as being members of one of two groups: "business firms" that hold as assets the physical means of production for the economy and finance their production decisions by issuing financial claims or securities; and "financial intermediaries" that hold financial claims as assets and finance these assets by issuing securities. Individuals or households are assumed to invest primarily in securities, and therefore invest only indirectly in physical assets. The markets in which these securities are traded are *capital markets*.
>
> The natural starting point for the development of investment theory is to derive the investment behavior of the individuals. It is traditional in economic theory to take the existence of households and their tastes as exogenous to the theory. However, this tradition does not extend to economic organizations and institutions. They are regarded as existing primarily because of the functions they serve instead of functioning primarily because they exist. Economic organizations are endogenous to the theory. To derive the functions of these economic organizations, therefore, the investment behavior of individuals must be derived first.

Thus we, too, begin with a community of natural individuals who engage in both financial and physical investment (and carry out production) in order to show how the separation of ownership and control emerges, with firms having the function of maximizing shareholders wealth[29]. As a consequence of adding production to the model, the one-date budget constraints (1.24) and (1.25) are modified to yield[30]

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[29] This approach to the understanding of capitalism is contestable on several grounds—but these issues are well beyond the scope of these Notes. For a critical appraisal of the Fisherian foundations of finance, see, for example, O'Sullivan (2000); and Ito and Lapavitsas (1999) for a Marxian perspective.

[30] Since the rate of depreciation of physical capital is 100%, the volume of investment or amount of capital goods purchased at date 0 coincides with the level $k$ of the capital input at that date. Accordingly, we shall use the same symbol, $k$, for both physical capital and productive investment.

$$c_0 = \omega_0 - b - k, \tag{3.3}$$

$$c_1 = \omega_1 + (1+r)b + y_1. \tag{3.4}$$

The constraint (3.3) states that endowed income at date 0 (a source of funds) must balance the sum of three uses of funds at that date: consumption, purchase of bonds, and investment in physical capital (or purchase of capital goods). If $b$ is negative, then $-b$ is a source of funds at date 0, and $-(1+r)b$ becomes a use of funds at date 1. It follows from (3.3) that the current saving of the agent is no longer synonymous to financial investment [as in (1.29)]; it is the algebraic sum of financial and physical investment:

$$s := \omega_0 - c_0 = b + k. \tag{3.5}$$

The date 1 income $y_1$ in (3.4) is the output that results from the physical investment $k$ at date 0, and hence it is related to that investment by the production function (3.1) of the agent.

The consumption-saving decision (i.e. the agent's decision on how to allocate her current income $\omega_0$ between consumption $c_0$ and savings $s$) is now interwoven with the "*portfolio selection" choice* where she decides how to allocate her savings between the two available investment opportunities—between the purchase of bonds and capital goods. In effect, the problem of choosing the best attainable consumption plan boils down to finding a portfolio $(b,k)$ of financial and physical investment that maximizes utility $U = U(c_0, c_1)$, viewed as a **composite function** of $b$ and $k$ on account of the constraints (3.3) and (3.4). The **first-order conditions** for an interior solution to this maximization problem read[31]

$$\frac{\partial U}{\partial b} = 0, \tag{3.6}$$

$$\frac{\partial U}{\partial k} = 0. \tag{3.7}$$

To express the partial derivatives in (3.6) and (3.7) in terms of marginal utilities and the marginal product of capital we need to invoke once again the pertinent chain rule of differentiation [Sydsaeter and Hammond (2006, Section 12.2)]. Thus conditions (3.6) and (3.7) read

[31] See, for example, Sydsaeter and Hammond (2006, Section 13.1). Our assumptions regarding preferences and production technology (including the premise about the asymptotic behavior of the indifference curves and the fulfillment of the Inada conditions by the neoclassical production function) ensure an interior solution—the maximizing values of $b$ and $k$ are in the interior of their admissible domains and yield a positive consumption stream. These assumptions also ensure that (3.6) and (3.7) yield a unique maximum for any given $r > -1$, as it should become clear from the forthcoming geometry of the optimal solution.

$$\frac{\partial U}{\partial b} = \frac{\partial U}{\partial c_0}\frac{\partial c_0}{\partial b} + \frac{\partial U}{\partial c_1}\frac{\partial c_1}{\partial b} = 0, \tag{3.8}$$

$$\frac{\partial U}{\partial k} = \frac{\partial U}{\partial c_0}\frac{\partial c_0}{\partial k} + \frac{\partial U}{\partial c_1}\frac{\partial c_1}{\partial k} = 0. \tag{3.9}$$

From (3.3) and (3.4) we derive $\partial c_0 / \partial b = -1$ and $\partial c_1 / \partial b = 1 + r$; $\partial c_0 / \partial k = -1$ and $\partial c_1 / \partial k = dy_1 / dk = f'(k)$. Hence (3.8) and (3.9) yield (3.10) and (3.11), respectively:

$$\frac{\partial U}{\partial c_0} = \frac{\partial U}{\partial c_1}(1 + r), \tag{3.10}$$

$$\frac{\partial U}{\partial c_0} = \frac{\partial U}{\partial c_1} f'(k). \tag{3.11}$$

Equation (3.10) is identical to the first-order condition (2.2). As already seen, it requires that the marginal loss in utility be equal to the marginal gain in utility from a one-unit marginal increase in financial investment. The new first-order condition (3.11) requires the same marginal balance in reference to physical investment: $\frac{\partial U}{\partial c_0}$ is the loss in utility if the agent purchases an additional unit of physical capital (and therefore foregoes the consumption of that unit) at date 0; $\frac{\partial U}{\partial c_1} f'(k)$ is the increase in utility she obtains from the extra $f'(k)$ units of consumption at date 1. The agent continues to invest in physical capital until the marginal loss equals the marginal gain. It cannot be overemphasized that the agent's optimal choice of portfolio selection $(b, k)$ requires that **both marginal conditions**, equations (3.10) and (3.11), must be satisfied simultaneously.

Overall, the utility-maximizing model of an agent's behavior now involves **four endogenous variables**: $b, k, c_0$, and $c_1$. The optimal values of these variables—often summarized as the 4-list $(b, k, c_0, c_1)$— are in principle determined by a system of **four** equations: the one-date budget constraints (3.3) and (3.4), and the first-order conditions (3.10) and (3.11) [for any given real interest $r > -1$ and agent characteristics $(U(\cdot), (\omega_0, \omega_1), f(\cdot))$]. We will rewrite this system in equivalent forms in the sequel so as to facilitate the geometrical interpretation of the constraints and the first-order conditions and to be able to decipher the properties of the solution (demand, supply, or net demand functions).

Begin with the first-order conditions: the system of (3.10) and (3.11) is equivalent to the system

$$MRS_{01}(c_0, c_1) = 1 + r, \tag{3.12}$$

$$f'(k) = 1 + r. \tag{3.13}$$

Equation (3.12) has been obtained from equation (3.10) by dividing both sides by $\frac{\partial U}{\partial c_1}$ and recalling the definition of $MRS_{01}(c_0, c_1)$ in (1.1). This equation is nothing other than the **tangency condition** (2.3); as we shall see soon, this is the tangency portrayed in Fig. 2.1 but the position (the horizontal intercept) of the intertemporal budget line is different. Equation (3.13) comes from the observation that the right-hand sides of (3.10) and (3.11) must be equal at the optimum (because the left-hand sides are identical),

$$\frac{\partial U}{\partial c_1} f'(k) = \frac{\partial U}{\partial c_1}(1 + r), \tag{3.14}$$

and the subsequent cancellation[32] of the common factor $\frac{\partial U}{\partial c_1}$. It is often referred to as the *optimal investment rule* [Williamson (2005, Ch. 9)]. Whether invested in bonds or in physical capital, a one-unit marginal sacrifice of current consumption entails the same utility loss, $\frac{\partial U}{\partial c_0}$; if invested in physical capital, $\frac{\partial U}{\partial c_1} f'(k)$ is the increase in utility the agent would obtain from the resulting extra $f'(k)$ units of consumption at date 1; if invested in bonds, $\frac{\partial U}{\partial c_1}(1 + r)$ is the increase in utility she would obtain from the extra payoff of $1 + r$ units of consumption at date 1. Equations (3.12) and (3.13) require—and equation (3.14) states so explicitly—that these two marginal gains in utility must be equal at the optimum. Equation (3.13) expresses the same condition in terms of future consumption (rather than in terms of utility). I n effect, (3.13) requires that the *gross real rate of returns* from productive and financial investments must get equal at the optimum.

To obtain a **gross rate of return**, we divide the payoff at date 1 from an investment at date 0 by the amount of that investment. Thus $1 + r$ is the gross (real) rate of return on bonds. If the payoff is not a linear function of the amount of investment (as in the case of intertemporal production), a **marginal** gross rate of return becomes the appropriate notion, namely the rate of change of the payoff at date 1 with respect to the volume of investment at date 0. To put it loosely, the marginal gross rate of return is the increase in the payoff at date 1 from a one-marginal increase in the amount of investment at date 0; so the (real) marginal gross rate of return from productive investment is the marginal product of capital, $f'(k)$.

When using the term 'rate of return' we typically mean **net rate of return**. To obtain a net rate of return, we subtract from the payoff at date 1 the amount of the associated investment at

[32] Recall that we anticipate a positive optimal consumption stream and that by assumption marginal utilities are positive at any positive consumption stream. Therefore, the division by $\partial U / \partial c_1$ is legitimate and the system of (3.12)-(3.13) is indeed equivalent to the system of (3.10)-(3.11).

date 0 before dividing by the amount of investment. Accordingly, the real rate of interest is the (net) real rate of return from investing in bonds (as we saw earlier), while the (net) *real rate of return from productive investment* is the *net marginal product of capital*, $f'(k)-1$. The net marginal product of capital is the marginal product of capital after taking into account the depreciation of capital, which here is by assumption 100%. We may restate the optimal investment rule (3.13) in the form

$$f'(k)-1=r \tag{3.15}$$

and interpret it as the requirement that the agent should invest in physical capital to the point where the real rates of return from investment in physical and financial assets (bonds) get equalized.

Turn next to the system of the two one-date constraints (3.3) and (3.4). From our experience with the model without production we want to identify the two equations that involve $c_0$ and $c_1$ as the only unknowns: the tangency condition (3.12) and the intertemporal budget constraint—appropriately modified to encompass the introduction of productive investment opportunities. To derive the later, solve (3.3) with respect to *b*,

$$b=\omega_0-c_0-k, \tag{3.16}$$

and insert (3.16) into (3.4); in turn, divide both sides of the resulting equation by 1+*r,* and rearrange terms to obtain

$$c_0+\frac{c_1}{1+r}=W_T, \tag{3.17}$$

where

$$W_T:=\omega_0+\frac{\omega_1}{1+r}-k+\frac{y_1}{1+r} \tag{3.18}$$

is the agent's *total wealth* at date 0. It consists of the endowed wealth $W_0:=\omega_0+\frac{\omega_1}{1+r}$ and the *wealth gain* associated with investment *k*, which is the *net present value*[33] *NPV* of the production plan $(-k, y_1=f(k))$:

$$NPV:=-k+\frac{y_1}{1+r}=-k+\frac{f(k)}{1+r}. \tag{3.19}$$

[33] In general terms, the net present value of an activity that involves benefits and costs distributed over time is the present value of the benefit stream minus the present value of the stream of costs. In the case of an investment project, its net preset value comes to be equal to the difference between the present value of the sequence of the future cash flows associated with the project and its current cost.

Note that $NPV$ is the intertemporal counterpart of the familial **profit expression** 'revenue minus cost' expressed in real terms; since the revenue $y_1$ is obtained one period after the incurrence of the current cost $k$, we subtract the latter from the present value $y_1/(1+r)$ of the revenue. Even though the total wealth $W_T$ in (3.17) involves the endogenous variable $k$, the optimal value of the latter is really fixed by the optimal investment rule (3.13) [as we shall explain shortly], so the system of (3.12) and (3.17) does in effect involve $c_0$ and $c_1$ as the only unknowns.

In sum, the system of the four equations—the one-date budget constraints (3.3) and (3.4), and the first-order conditions (3.10) and (3.11)—that determines the optimal values of $(b,k,c_0,c_1)$ [for any given real interest $r>-1$ and agent characteristics $(U(\cdot),(\omega_0,\omega_1),f(\cdot))$] is **equivalent** to the system of equations (3.12), (3.13), (3.16), and (3.17) [accompanied by the production function relation (3.1) and the definitional equation (3.18)].

## 4. The Fisher Separation Theorem

We now wish to examine how the agent's optimality conditions, i.e. the system of equations (3.12), (3.13), (3.16), and (3.17), determine the optimal values of the endogenous variables $(b,k,c_0,c_1)$ [for any given $r>-1$]. Especially, we want to demonstrate how such an analysis leads to the separation of production and consumption decisions or, equivalently, the separation of productive and financial investment decisions. The gist of this result, which is known as the *Fisher separation theorem*, is that real investment decision rules can be determined without knowledge of the details of the agent's personal preferences regarding alternative consumption streams. In the context of a private ownership economy where physical production is carried by business firms, the Fisher separation theorem entails that one can devise a set of real investment decision rules such that if the managers of a business firm follow these rules, the firm will operate "as if" the owners of the firm made the production decisions directly. Hence, unlike the utility function of an individual which is taken to be exogenous in the consumption-saving and portfolio selection problem, the objective function for production decisions by the firm is derived and therefore endogenous [ Merton (1982), p. 642]. We shall establish that this objective function is none other than the net present value of the income stream of the production unit.

Our starting point in the study of the optimality conditions is the optimal investment rule (3.13): it alone determines the optimal value of $k$ solely in terms of the production function and the interest rate, independently of the agent's utility function. To put it otherwise, two agents with very different attitudes regarding consuming now versus consuming later but with the same technological know-how (i.e. with identical production functions) would select the same volume of productive investment. On account of the neoclassical nature of the production function, equation (3.13) has a unique solution for any given $r>-1$, designated

$$k = k^*(r) \qquad (r>-1) \tag{4.1}$$

and referred to as the agent's *demand function for productive investment* (or *real investment demand function*). This can be seen either by solving equation (3.13) graphically[34] or by graphing (3.15) directly (with $r$ measured along the vertical axis). See Fig. 4.1. Clearly, the graphs of the direct investment demand function (4.1) and the inverse demand function $r = f'(k) - 1$ coincide[35]. Thus the optimal investment rule entails the existence of a downward-sloping demand curve for productive investment (reflecting a diminishing marginal product of capital), which is entirely independent of the personal preferences (or the intertemporal utility function) of the agent.

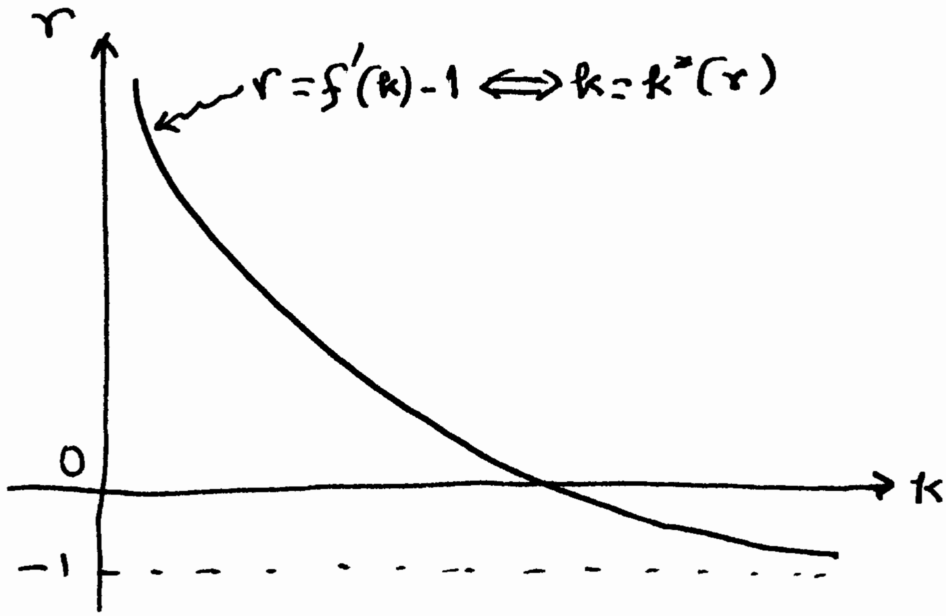


Fig. 4.1 Real Investment Demand

To get a sense of the investment demand function $k = k^*(r)$, consider the case of a specific class of neoclassical production functions

$$y_1 = k^{\alpha} \quad (0 < \alpha < 1). \tag{4.2}$$

[That is, $f(k) \equiv k^{\alpha}$.] The optimal investment rule (3.13) reads $\alpha k^{\alpha-1} = 1 + r$ or $\alpha = (1+r)k^{1-\alpha}$, which yields

$$k = \left(\frac{\alpha}{1+r}\right)^{\frac{1}{1-\alpha}}. \tag{4.3}$$

[34] The graphical solution should make clear the importance of the Inada condition $dy_1 / dk \to +\infty$ as $k \to 0$ (Footnote 27) for ensuring that (3.13) has a unique solution **for any** $r > -1$, however large.

[35] Technically, (4.1) is the inverse of (3.15) and, by the inverse function rule of differentiation, $dk / dr = 1 / f''(k) < 0$ for all $k > 0$ [Sydsaeter and Hammond (2006), Section 7.3].

In other words, $k^*(r) \equiv \left(\frac{\alpha}{1+r}\right)^{\frac{1}{1-\alpha}}$ for production functions of the form (4.2).

We wish to show next that the optimal investment rule (3.13) is equivalent to a **production decision rule**, which is independent of the agent's preferences and endowment. Returning to the system of equations (3.12), (3.13), (3.16), and (3.17), we sense from the preceding discussion that the overall utility- maximizing choice of $(b, k, c_0, c_1)$ subject to the constraints (3.16) and (3.17) amounts to two separate actions, reflecting two separate roles for the agent—the role of consumer/financial investor, on the one hand, and production manager, on the other. To separate out the agent's activity as a consumer/financial investor, imagine for a moment that the volume of investment $k$ is preset at a particular level. As a consequence, the levels of output and the total wealth $W_T$ of the agent are fixed as well. Then the first-order condition for maximizing $U(c_0, c_1)$ subject to (3.17) is the tangency condition (3.12); this is the tangency illustrated in Fig. 2.1, with $W_0$ being replaced by the preset value $W_T$ of total wealth. The utility-maximizing values of $c_0$ and $c_1$ **for given** $k$ are fully determined by the system of equations (3.12) and (3.17), while the associated value of $b$ can be calculated from equation (3.16). The solution values of $c_0$ and $c_1$ are exactly the functions in (2.8), with $W_0$ being replaced by $W_T$ [as defined by equation (3.18)]:

$$c_0 = \bar{c}_0(1+r, W_T) \text{ and } c_1 = \bar{c}_1(1+r, W_T). \tag{4.4}$$

But the level of utility realized in this fashion [i.e. the value of the utility function at the consumption stream (4.4)] is not necessarily the maximum attainable. As shown in Section 2, **more wealth is better.** If the agent could raise her total wealth by varying the volume of real investment, she would be able to shift out the intertemporal budget line and attain a greater level of maximum utility. Therefore, in order to attain the maximum possible level of utility, the agent should choose the amount of real investment $k$ so as to maximize her total wealth (3.18) or, equivalently, the net present value (3.19). In other words, in her role as a production manager, the agent should follow the **net present value rule**: invest (choose $k$) so as to maximize the net present value of the investment. In view of (3.19), the first order condition for $NPV$ maximization reads

$$\frac{dNPV}{dk} = -1 + \frac{1}{1+r}\frac{dy_1}{dk} = -1 + \frac{f'(k)}{1+r} = 0, \tag{4.5}$$

which is equivalent to the optimal investment rule (3.13) for all $r > -1$, and hence (4.1) is the maximizer of the agent's $NPV$. Equations (3.13) and (4.5) are two equivalent ways to express the familiar 'marginal revenue equals marginal input cost' condition for profit (net present value) maximization. Thus by substituting (4.1) into (3.18) we obtain the agent's maximum attainable wealth[36] $W_T^*$ (for any given interest rate $r > -1$):

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[36] The first-order condition yields a global maximum because the graph of $NPV$ as a function of $k$ [equation (3.19)] is strictly concave—the second derivative $d^2NPV / dk^2 = f''(k)/(1+r)$ is negative

$$W_T^* := \omega_0 + \frac{\omega_1}{1+r} - k^*(r) + \frac{y_1^*(r)}{1+r}, \tag{4.6}$$

where

$$y_1 = y_1^*(r) := f(k^*(r)) \tag{4.7}$$

is the agent's *output supply function.* It is easy to show that $y_1 = y_1^*(r)$ is a strictly decreasing function. To illustrate, if the production function is of the form (4.2), then, on account of (4.3), we have

$$y_1 = \left(\frac{\alpha}{1+r}\right)^{\frac{\alpha}{1-\alpha}} \equiv y_1^*(r). \tag{4.8}$$

In view of the preceding arguments regarding the structure of the optimality conditions, it is natural to envision a two-stage process for solving the problem of maximizing intertemporal utility subject to the one-period budget constraints (3.3) and (3.4)—namely, for solving the system of equations (3.12), (3.13), (3.16), and (3.17). In **Stage I**, we determine the optimal level of productive investment by maximizing its net present value (and, as a consequence, the agent's total wealth), yielding the optimal investment rule (3.13). The resulting optimal investment (4.1) and the associated maximum attainable wealth (4.6) are entirely independent of the agent's utility function. In **Stage II,** we find the optimal consumption stream by maximizing the agent's intertemporal utility subject to the lifetime budget constraint corresponding to the maximum total wealth determined in Stage I, namely subject to equation (3.17) with $W_T = W_T^*$ :

$$c_0 + \frac{c_1}{1+r} = W_T^*. \tag{4.9}$$

Geometrically, the solution of the system of equations (3.12) and (4.90) is the point of tangency depicted in Fig. 2.1 but with the position of the intertemporal budget line fixed at $W_T^*$ instead of $W_0$. See Fig. 4.2. Once again, starred symbols are abbreviations for optimal values associated with a given but unspecified *r*: $(b^*, k^*, c_0^*, c_1^*) := (b^*(r), k^*(r), c_0^*(r), c_1^*(r))$. The demand functions (2.4) in the presence of productive opportunities are obtained from the expressions (4.4) by substituting $W_T$ for its optimal value (4.6):

$$c_0 = c_0^*(r) := \bar{c}_0(1+r, W_T^*) \text{ and } c_1 = c_1^*(r) := \bar{c}_1(1+r, W_T^*) \qquad (r > -1). \tag{4.10}$$

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for all positive values of *k* on account of diminishing marginal productivity. A boundary maximum (*k*=0) is precluded because *NPV* is strictly increasing for $0 \le k < k^*(r)$. Indeed, the first derivative of *NPV* is positive for $0 < k < k^*(r)$ and NPV itself is a continuous function of *k* for all $k \ge 0$. See Sydsaeter and Hammond (2006), Sections 6.3 and 8.4.

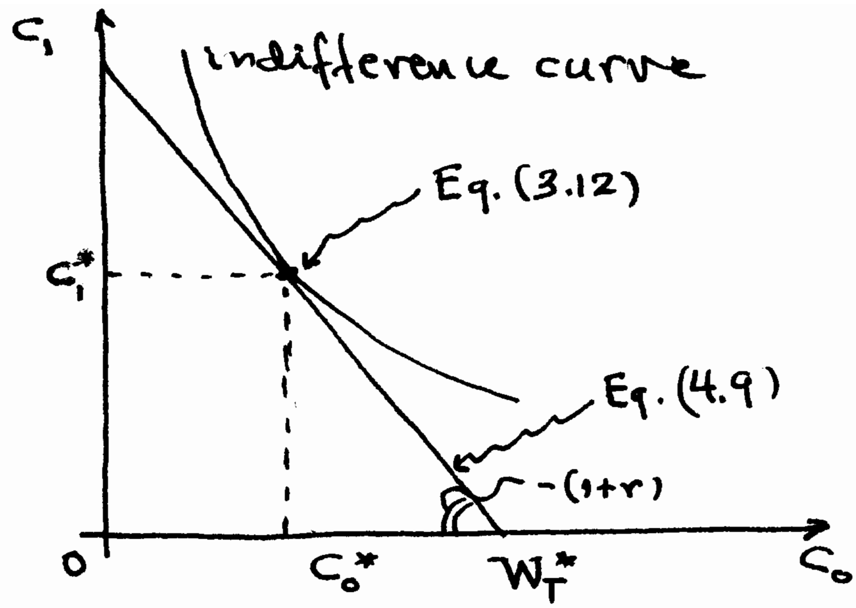


Fig. 4.2 Solution of the System of Equations (3.12) and (4.9)

The net demand function for bonds is obtained from (3.16) by using the expressions for $k$ and $c_0$ in (4.1) and (4.10), respectively:

$$b = \omega_0 - c_0^*(r) - k^*(r) =: b^*(r) \qquad (r > -1). \qquad (4.11)$$

In turn, insert $c_0$ from (4.10) into (2.6) to derive the saving function $s = \omega_0 \text{-} c_0^*(r) =: s^*(r)$; it is important to note that the identity $s^*(r) \equiv b^*(r)$ **is no longer true;** instead, in view of (3.5), we now have the identity

$$s^*(r) \equiv b^*(r) + k^*(r). \qquad (4.12)$$

The two-step solution procedure just described is the essence of the *Fisher separation theorem,* which is therefore formulated as follows[37]:

> Given perfectly competitive markets, an individual agent finds her utility maximizing combination of physical and financial assets by applying a two-stage decision rule. **First**, the agent determines the level of productive investment that maximizes net present value (and her total wealth)—the optimal investment rule. The optimal amount of productive investment and the associated maximum wealth are entirely independent of personal preferences. **Second**, the agent determines her optimal consumption plan and her optimal volume of financial assets (or liabilities) by maximizing her intertemporal utility subject to the intertemporal constraint of her total wealth, established in the first stage by means of the optimal investment rule.

[37] Irvin Fisher was the first economist to see the theoretical concept of 'separation' as described above and the practical implications. See Broll and Milde (1999, pp. 310-314), Copeland and Weston (1992, pp. 3-20), and Hirshleifer and Hirshleifer (1998, Section 14.4) for more detailed discussion.

As a consequence of the separation theorem, the owners of any given production unit may delegate the running of their enterprise to professional managers with a jointly agreed directive: the net present value rule. Broll and Milde (1999, p. 310) summarize this point very nicely:

> The two sub-decisions are determined by completely different motives. The volume of financial assets is dominated by personal preferences of the investor, the volume of real assets is not. The feature of separation has two important implications: first, the delegation of decision- making authority to professional managers is possible; second, the observable fact of separation of ownership and control in modern corporations has a solid theoretical foundation.

It stands to reason that the Fisher separation theorem should hold more broadly in situations where the spectrum of real investment projects is discrete rather than continuous. Constrained utility maximization would still require the selection of the real investment projects that maximize net present value the—the net present value rule—but (3.13) would no longer constitute the pertinent optimal investment rule[38].

It remains to examine how the agent's real saving and net demand for bonds respond to changes in the real interest rate. Do our conclusions in a pure exchange environment (Section 2) carry over in a financial economy with productive investment opportunities? We take up these issues along with the geometry of the separation theorem in the next section.

## 5. The Geometry of the Separation Theorem

The separation of production and consumption decisions is best illustrated with the aid of the *intertemporal production possibility frontier* (PPF). The PPF is the technologically attainable output streams $(x_0, x_1)$ on account of the agent's production function $y_1 = f(k)$ and endowment $(\omega_0, \omega_1)$. Clearly, a point in the PPF is an attainable consumption stream under autarky (i.e. without issuing or purchasing bonds). Accordingly, we may find a parametric representation of the PPF by setting $b = 0$ and $(c_0, c_1) = (x_0, x_1)$ in (3.3) and (3.4):

$$x_0 = \omega_0 - k, \tag{5.1}$$

$$x_1 = \omega_1 + y_1. \tag{5.2}$$

Recall that the variable $k$ ranges over nonnegative values and is related to $y_1$ via the agent's production

$$y_1 = f(k). \tag{3.1}$$

[38] See, for example, Hirshleifer and Hirshleifer (1998), Section 14.4. In this reference, the net present value rule is referred to as 'the present-value rule'.

We may also describe the PPF by a single equation by eliminating $k$ from (5.1) and (5.2): Solve (5.1) for $k$, $k = \omega_0 - x_0$, and insert into (3.1) and (5.2) to obtain

$$x_1 = \omega_1 + f(\omega_0 - x_0) \qquad (x_0 \leq \omega_0). \tag{5.3}$$

The pair of equations (5.1)-(5.2), on the one hand, and the single equation (5.3), on the other, are just two alternative ways of describing the PPF. It is clear from (5.3) that the shape of the PPF reflects the properties of the neoclassical production function (as portrayed in Fig. 2.1). The graph of the production function (3.1) in the broken-line coordinate axes in Fig. 5.1 is the PPF (i.e. the graph (5.3)) in the $Ox_0x_1$ coordinate system. Thus the PPF slopes downward because of positive marginal productivity; it is strictly concave because the neoclassical production exhibits diminishing marginal productivity. Strict concavity here is tantamount to a rising *marginal rate of intertemporal transformation* ($MRT_{01}$), defined as the absolute value of the slope of the PPF:

$$MRT_{01}(x_0) := -\frac{dx_1}{dx_0} = -\frac{dx_1 / dk}{dx_0 / dk} = f'(k) = f'(\omega_0 - x_0) \qquad (x_0 < \omega_0). \tag{5.4}$$

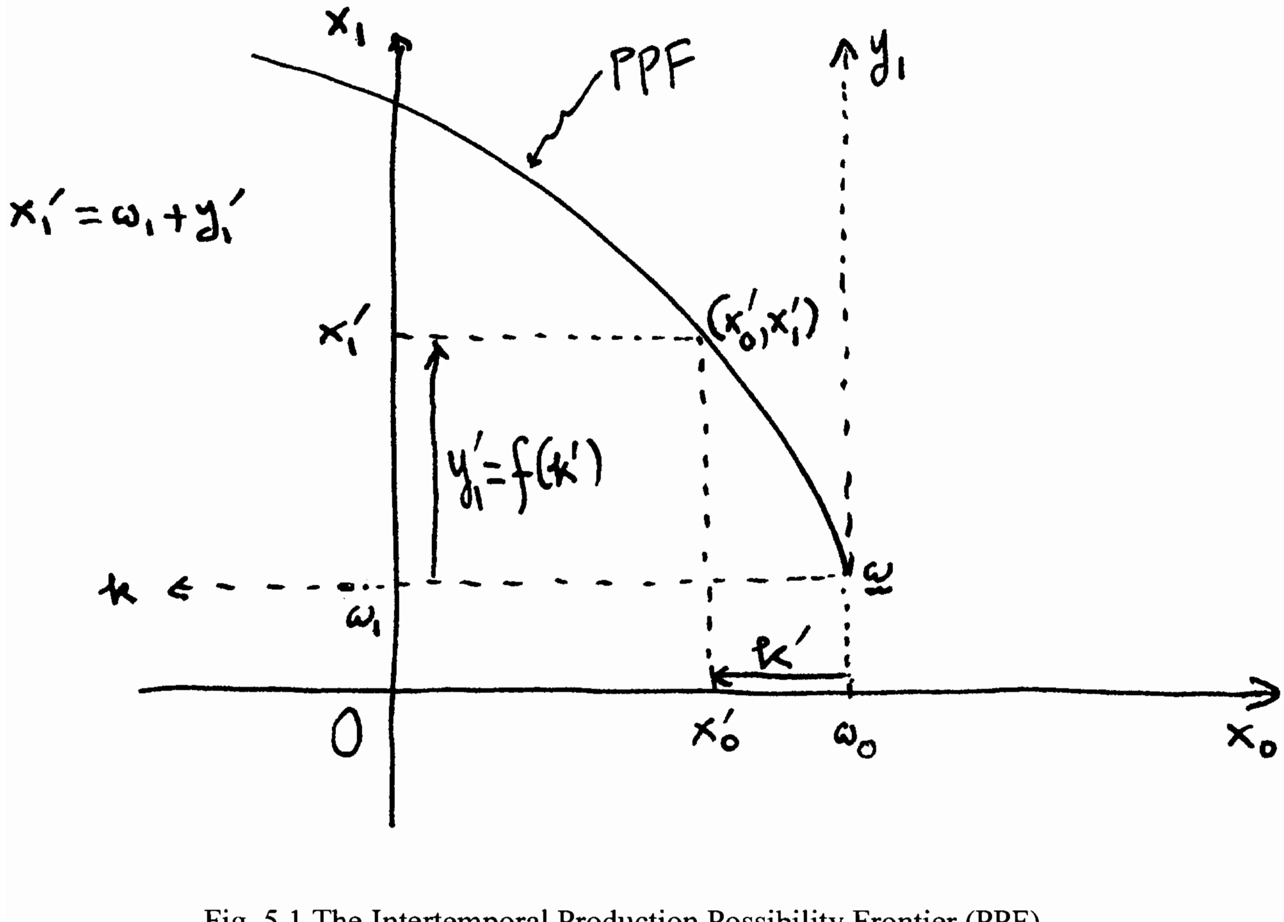


Fig. 5.1 The Intertemporal Production Possibility Frontier (PPF)

In view of (5.1) and (5.2), the total wealth (3.18) of the agent can be expressed in terms of the point $(x_0, x_1)$ in the PPF that corresponds to productive investment *k*:

$$W_T = x_0 + \frac{x_1}{1+r}. \tag{5.5}$$

The net present value rule, which is tantamount to maximizing the agent's wealth (3.18) over $k \geq 0$, translates to the maximization of (5.5) over $(x_0, x_1)$ in the PPF: Choose $(x_0, x_1)$ to

$$\max_{x_0, x_1} W_T \text{ subject to (5.3).} \tag{5.6}$$

Alternatively, wc may view (5.5) as a composite function of *k* on account of (5.1)-(5.2). Hence we may convert (5.6) to the unconstrained optimization problem[39]

$$\max_{k} W_T. \tag{5.7}$$

The first-order condition for (5.7) reads

$$\begin{aligned} \frac{dW_T}{dk} &= \frac{\partial W_T}{\partial x_0}\frac{dx_0}{dk} + \frac{\partial W_T}{\partial x_1}\frac{dx_1}{dk} \\ &= -1 + \frac{1}{1+r} f'(k) \\ &=0, \end{aligned} \tag{5.8}$$

which is the optimal investment rule (3.12), as expected. In view of (5.4), (5.8) yields

$$MRT(x_0) = 1 + r. \tag{5.9}$$

Thus the optimal investment rule amounts to finding a point on the PPF at which an **iso-wealth line**

$$x_0 + \frac{x_1}{1+r} = c \tag{5.10}$$

is tangent to the PPF. [Here *c* stands for some constant level of wealth.]

Accordingly, one may solve problem (5.6) geometrically as follows. Draw an iso-wealth line (5.10) whose slope is $-(1+r)$ in the $Ox_0x_1$ coordinate system; while holding this slope constant, keep moving the iso-wealth line northeast to the position where even the slightest

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[39] We have already established in Footnote 37 that a boundary optimum $(k = 0)$ is precluded and that the first order condition does yield a global maximum for NPV and total wealth.

further outward move will take us out of the area bounded by the PPF. This is the position where the iso-wealth line is tangent to the PPF, with its horizontal intercept measuring the agent's maximum total wealth $W_T^*$ (for given $r$). Since the maximum wealth line is the agent's intertemporal budget line, we may now portray the production and consumption optima in the same diagram[40]. See Fig. 5.2. From Fig. 5.2 we see once again that the *production optimum,* i.e. the optimal choice $k^*$ of productive investment, is independent of the intertemporal preferences

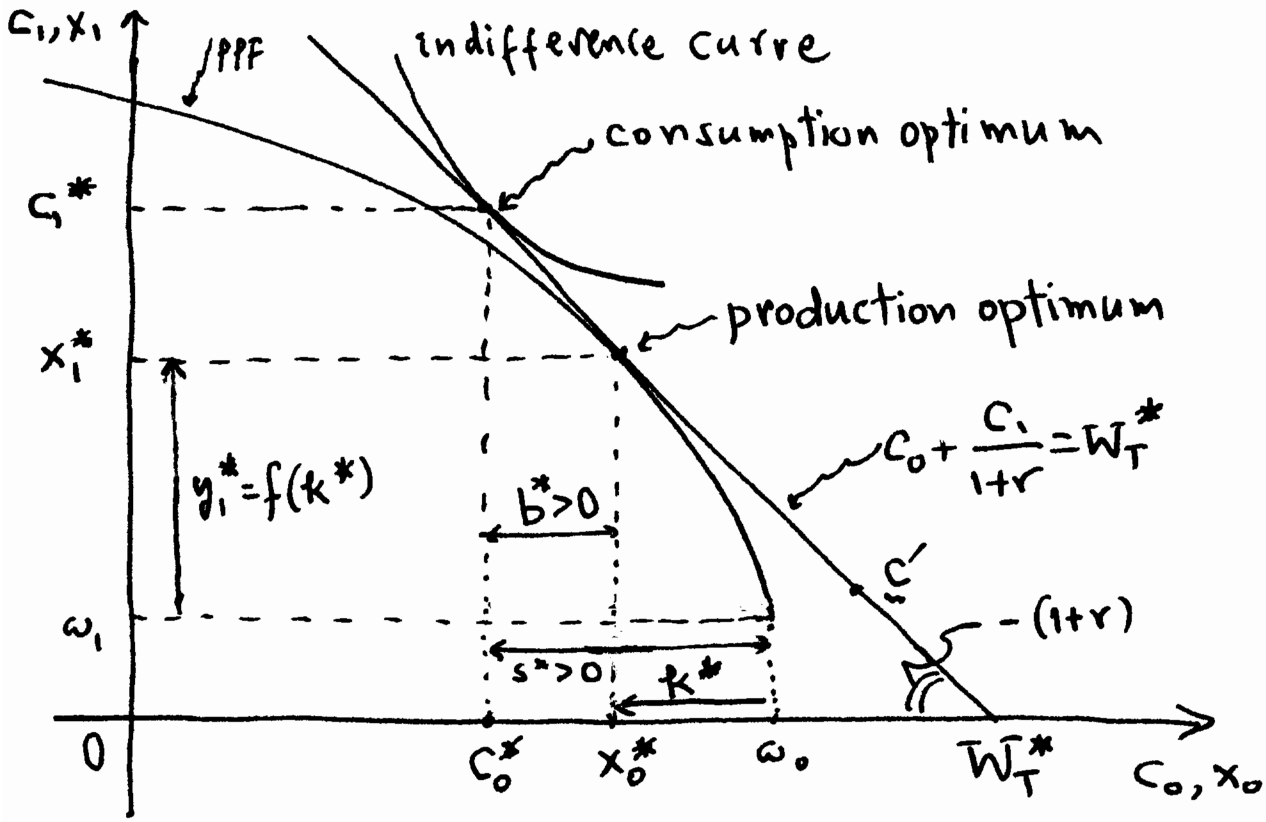



Fig 5.2 The Double-Tangency Solution

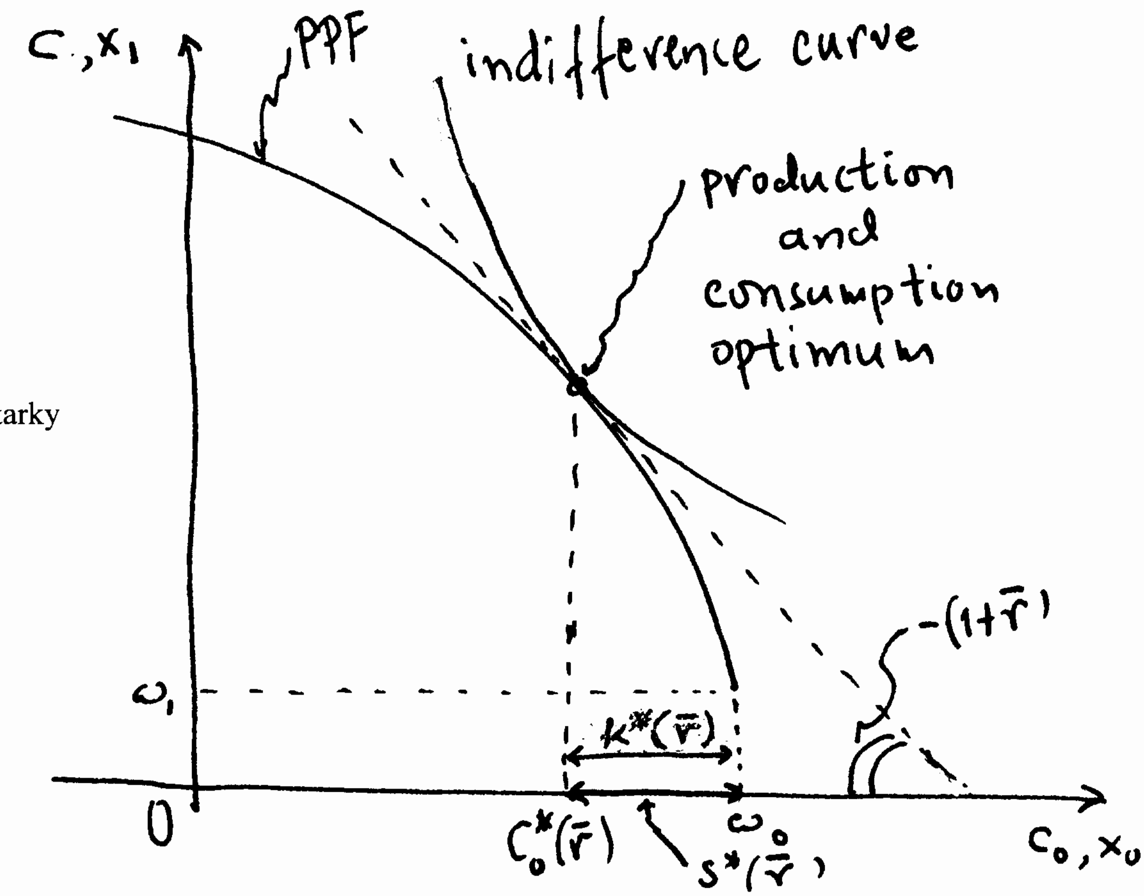
of the agent. But the optimal level of financial assets $b^*$ and the associated *consumption optimum* $(c_0^*, c_1^*)$ **do** depend on the intertemporal preferences of the agent. In Fig. 5.2, we portray a case where the individual chooses to save the amount $\omega_0 - c_0^* > 0$ and allocate it between the purchase of capital goods $k^*$ and financial assets $b^*$ as indicated. Less impatient individuals would have saved even more. More impatient agents would save less. For example, a highly impatient agent would choose a consumption optimum like $c'$, entailing dissaving; the sum of her productive investment and current consumption would exceed her endowed income $\omega_0$—the excess would have to be financed by borrowing (i.e. by issuing bonds).

[40] See Hirshleifer & Hirshleifer (1998, Ch. 14) and Copeland & Weston (1992, pp.1-20) for a more detailed discussion of diagrams of this sort, and analysis via the intertemporal PPF.

Once again, the benchmark interest rate—the autarky or no-borrowing, no-lending interest rate $\bar{r}$ — is important for ascertaining whether the agent would choose to become a borrower or lender at any give interest rate. But now $\bar{r}$ is no longer determined by (2.19) or (2.20). In autarky, the agent is self-financing her desired level of productive investment out of her savings. See Fig. 5.3. In other words, the autarky interest rate is that value $\bar{r}$ of the interest rate at which the agent's production and consumption optima coincide. From (4.11) we infer that $\bar{r}$ must satisfy

$$b^*(\bar{r}) = 0 \text{ or } s^*(\bar{r}) = k^*(\bar{r}). \tag{5.11}$$

Fig. 5.3 Autarky




With some plausible restrictions on the relative sizes of the substitution and wealth effects, the arguments that led to the saving curve in Fig. 2.3 still apply (though in a somewhat more complex fashion). You may presume, therefore, that the graph of $s = s^*(r)$ looks pretty much like the saving curve in Fig. 2.3. Combine this proposition with the investment demand curve in Fig. 4.1, and recall the identity (4.12) along with condition (5.11) to obtain Fig. 5.4.

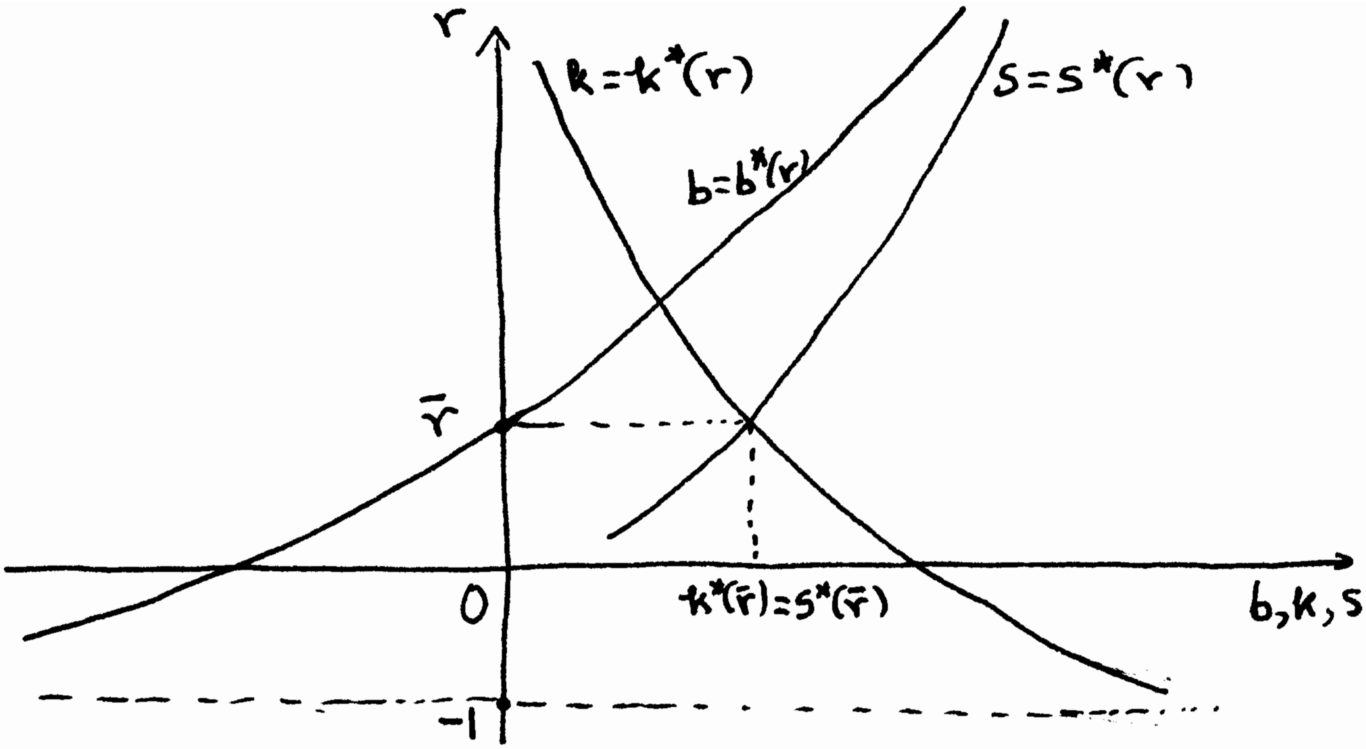



Fig 5.4 Graphs of the functions $b = b^*(r), k = k^*(r)$, and $s = s^*(r)$

**6. General Equilibrium**

Thus far we have discussed the behavior of a single agent, faced with a given interest rate. The next question is how the (real) interest is determined. To do so, we need to aggregate individual actions to obtain aggregate demand/supply functions and write out market-clearing conditions. The economy is in *general equilibrium* if demands and supplies balance in each of the three markets of the model. Unlike the equilibria in the markets for bonds and goods at date 0, the equilibrium in the goods market at date 1 is not currently observed, but it is (by assumption) correctly anticipated.

Use capital letters to denote aggregate quantities, and superscripts for agent-level variables, supposing that the $N$ agents that constitute the economy are labeled $1,2,\ldots,N$. Begin with aggregate consumption demand and the economy's total endowment of goods at dates 0 and 1:

$$C_t := c_t^1 + c_t^2 + \cdots + c_t^N \qquad (t = 0,1), \tag{6.1}$$

$$\Omega_t := \omega_t^1 + \omega_t^2 + \cdots + \omega_t^N \quad (t = 0,1). \tag{6.2}$$

Recall next that physical capital is like seed or material that is completely used up (or fully depreciated) during the production period (from date 0 to date 1). Accordingly, aggregate capital input $K$ and aggregate productive investment $I$ are identically equal at date 0:

$$I \equiv K := k^1 + k^2 + \cdots + k^N. \qquad (6.3)$$

The aggregate supply of goods at date 1, $Y_1$, is linked to individual supplies by

$$Y_1 := y_1^1 + y_1^2 + \cdots + y_1^N. \qquad (6.4)$$

Finally, the aggregation of the individual net demands for bonds yields the aggregate *excess demand* for bonds,

$$B := b^1 + b^2 + \cdots + b^N. \qquad (6.5)$$

Thus if $B > 0$, then the total demand for bonds exceeds total supply (or the supply of lending exceeds the demand for borrowing). On the other hand, a situation where $B < 0$ would mean that the demand for borrowing exceeds the supply of lending. Hence the market clearing condition for bonds should require that $B = 0$.

Corresponding to the functions (4.1), (4.10), and (4.11) we have the aggregate demand/excess demand functions[41]

$$I = I(r) \equiv K(r); C_0 = C_0(r); C_1 = C_1(r); B = B(r). \qquad (6.6)$$

Further, from (4.7) we obtain the aggregate output supply function of at date 1:

$$Y_1 = Y_1(r). \qquad (6.7)$$

The **total supply** of goods at date 1 is the sum $\Omega_1 + Y_1$ of the total endowment $(\Omega_1)$ and aggregate output $(Y_1)$ of goods at that date.

The equilibrium or market-clearing condition for the goods market at date 0 reads

$$C_0(r) + I(r) = \Omega_0. \qquad (6.8)$$

Likewise, the equilibrium condition for the goods market at date 1 is

$$C_1(r) = \Omega_1 + Y_1(r). \qquad (6.9)$$

Finally, we have the market-clearing condition for the bond market:

$$B(r) = 0. \qquad (6.10)$$

[41] We shall not carry over the star superscript in the symbols of aggregate demand, aggregate supply, or aggregate excess demand functions.

It seems we have three equations, (6.8)-(6.10), that involve only one unknown, $r$. However, these equations are not independent. To see this, aggregate the one-period budget constraints (3.3) and (3.4) over all agents to obtain

$$C_0(r) \equiv \Omega_0 - B(r) - I(r), \tag{6.11}$$

$$C_1(r) \equiv \Omega_1 - (1+r)B(r) + Y_1(r). \tag{6.12}$$

The triple-line equality sign underscores the fact that (6.11) and (6.12) must hold true identically, i.e. for all $r > -1$ (because the individual demand/supply/net demand functions (4.1), (4.7), (4.10), and (4.11) must satisfy (3.3) and (3.4) identically). Hence if $r = \hat{r}$ satisfies (6.10), that is, if

$$B(\hat{r}) = 0, \tag{6.13}$$

then it follows from (6.11) and (6.12) that $r = \hat{r}$ satisfies (6.8) and (6.9) as well. In fact, if any of the three equilibrium conditions (6.8)-(6.10) is satisfied at a certain interest rate, so do the other two on account of (6.11)-(6.12). This is an example of **Walras' law**: only one of the market-clearing conditions is independent[42]. In principle, this independent equation determines the *equilibrium real interest rate*. If, for instance, we choose to work with (6.11), then $\hat{r}$ [as specified by equation (6.13)] is the equilibrium real interest rate. All individual actions are in turn determined by setting $r = \hat{r}$ in (4.1), (4.10), and (4.11) for each agent.

With the agent-level curves as given in Fig. 5.4, equation (6.10) may indeed have a unique solution, $\hat{r} > 0$. See Fig. 6.1.

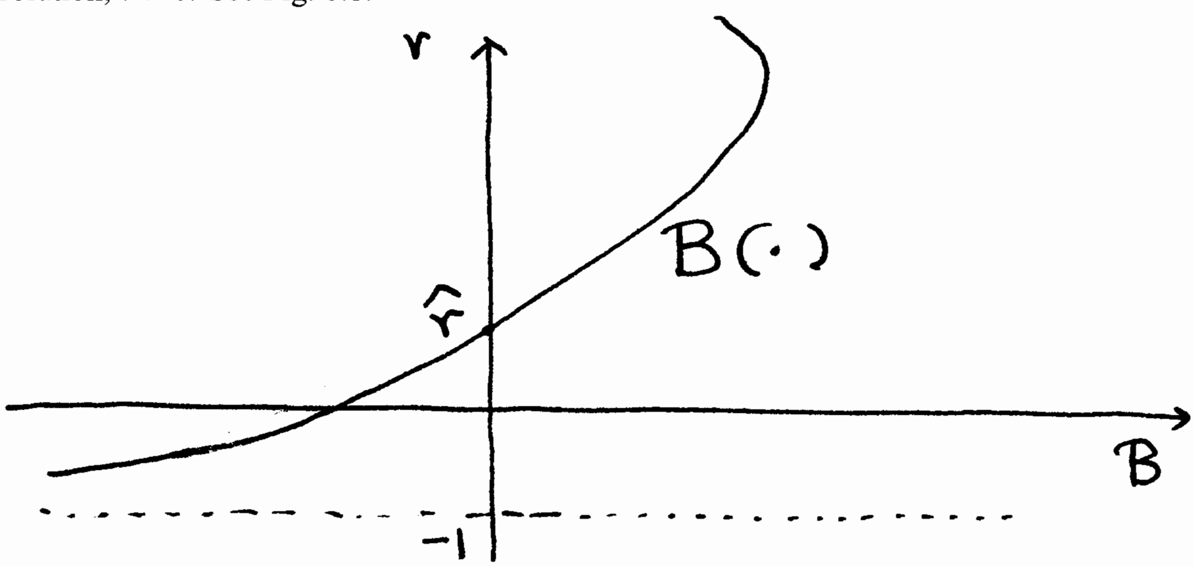



Fig. 6.1 The Equilibrium Real Interest Rate

[42] Strictly speaking, the presence of two one-period budget constraints entails **two** Walras' laws, the aggregate budget identities (6.11) and (6.12).

The positive slope $B'(\hat{r}) > 0$, ensures that the equilibrium is (locally) stable [Burmeister (1980), p. 84]. Loosely speaking, if $r > \hat{r}$, then there is a positive excess demand for bonds, exercising an upward pressure on the price of bonds and therefore a downward pressure on the interest rate. An interest rate somewhat below $\hat{r}$ would generate a negative excess demand or positive excess supply of bonds, putting a downward pressure on the price of bonds and hence an upward pressure on the interest rate.

Instead of focusing on the bond market, one could choose to work with the goods market at date 0: the equilibrium equation (6.8). Since

$$S(r) := \Omega_0 - C_0(r) \tag{6.14}$$

is the aggregate saving function, we may rewrite (6.8) in the familiar form

$$I(r) = S(r). \tag{6.15}$$

Clearly, the equilibrium interest rate $\hat{r}$ can be found by solving (6.15) instead of (6.10).

To get a sense of the state of general equilibrium and some of its properties, consider the special case of a pure exchange economy with just two agents, called agent 1 and agent 2. To justify the assumption of perfect competition, we think of agents 1 and 2 as the representatives of two large groups of identical individuals. Recalling the shape of the net demand curve of each agent in Fig. 2.3, we may depict the equilibrium interest rate in 6.2. Since $-\hat{b}^1 = \hat{b}^2$ at $\hat{r}$, $B(\hat{r}) = \hat{b}^1 + \hat{b}^2 = 0$—the equilibrium condition (6.13) is indeed satisfied.

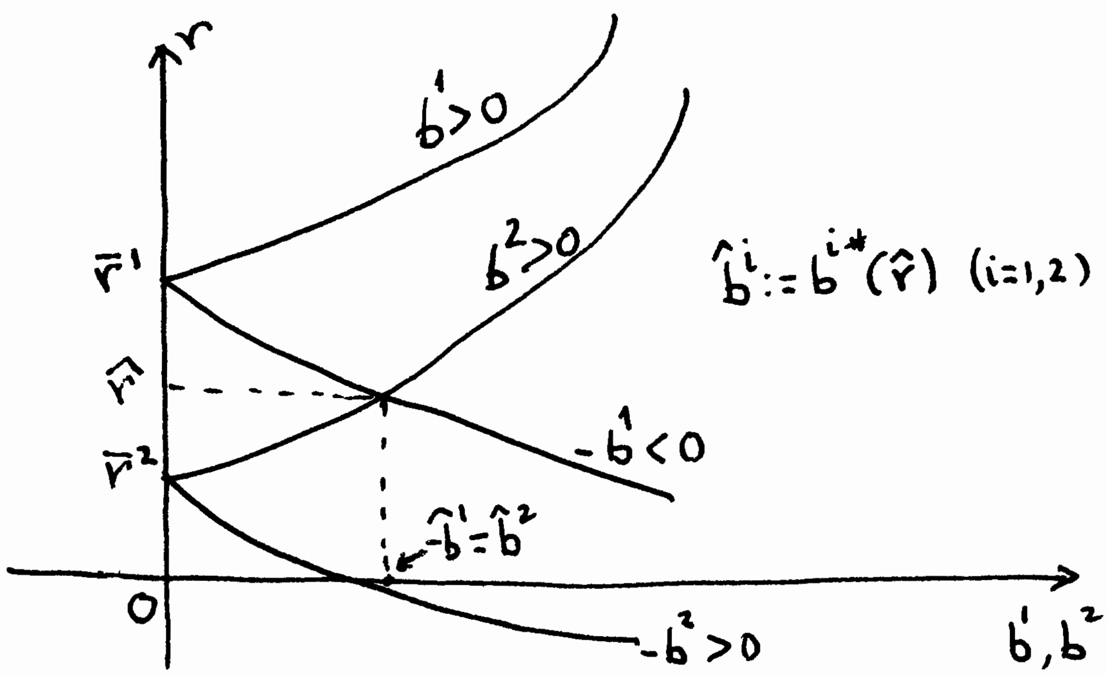



Fig. 6.2 Equilibrium in a Two-agent Economy

In the equilibrium portrayed in Fig. 6.2 agent 1 is a borrower, while agent 2 is a lender. It would be the other way around if $\bar{r}^1 < \bar{r}^2$ instead of $\bar{r}^1 > \bar{r}^2$. The crucial point for the formation of a bond market is the heterogeneity of the agents, expressed in terms of different autarky interest rates, that is, different marginal rates of substitution of consumption now for consumption later at their respective endowed output streams. For in this case both agents would gain by exchanging current for future goods, i.e. by trading bonds[43]. The purchase and sale of bonds will cease in equilibrium because the marginal rates of substitution will get equalized, eliminating the possibility of gaining from additional trading. This a general conclusion that holds true for an arbitrary number of heterogeneous agents: while their characteristics and choices of portfolios and consumption plans may vary wildly, they all optimize in equilibrium and hence they equate their marginal rates of substitution to the same equilibrium (gross) interest rate as dictated by the optimality condition (2.3)[44]. To put it otherwise, the equilibrium consumption streams of the agents constitute a **Pareto optimal** intertemporal allocation of resources[45]: there is no feasible reallocation of the aggregate endowments among agents that would raise the utility of some agent without decreasing the utility of somebody else. It is worth noticing that one plus the equilibrium real interest rate expresses the **common marginal valuation** of current consumption (in terms of future consumption) by all agents.

If production is added to the model, then the marginal product of capital (or the marginal rate of transformation of current goods to future output) gets equalized across producers in equilibrium as well on account of the first-order condition (3.13). [The marginal rates of substitution get equalized because of the first-order condition (3.12)]. So one plus the equilibrium interest rate also expresses the **common marginal cost** of current consumption (in terms of future consumption) of all agents. In sum, an equilibrium collection of individual portfolios and consumption plans amounts to a Pareto optimal intertemporal allocation of resources because (a) individual constrained optimization entails individually feasible actions and the equalization of marginal rates of substitution and marginal rates of transformation across agents (along with the equality of their common marginal valuation and common marginal cost); and (b) the balance of demand and supply in all markets ensures the global feasibility (or aggregate consistency) of individual choices.

## 7. Equilibrium and Comparative Statics with the Device of Representative Agent

The general equilibrium model of the preceding section captures Fisher's vision of linking interest, saving and investment to 'time preference' (intertemporal preferences), 'time endowment' (endowed real income stream), and 'time productivity' (intertemporal production function). But these fundamentals vary across the numerous individuals that constitute an economy, making the general equilibrium model too complex to determine the impact of a change in the fundamentals on the equilibrium interest rate as well as on the equilibrium levels of aggregate investment. To avoid the complications that result from the heterogeneity of agents, it

[43] This can be illustrated very nicely with the aid of an **Edgeworth box** [Varian (2006), Ch. 31].
[44] Compare with Varian's discussion in Section 5.5.
[45] See Varian (2006), Chapters 31 and 32, for a more discussion of the concept of Pareto optimality.

has become customary (and shown to be fruitful especially in macroeconomics and finance) to assume that all agents are identical, so that their collective behavior can be modeled by the behavior of a **representative agent**. Once this assumption is made, the individual-level analysis of Sections 4 and 5 applies to the whole market, as though the market were a single decision-maker[46]. Individual quantity variables such as consumption, endowed income, output, wealth, etc. are reinterpreted as the respective **quantities per capita**[47].

The intertemporal utility function $U = U(c_0, c_1)$ of the representative agent is usually assumed to be of the time-additive form

$$U(c_0, c_1) := u(c_0) + \delta u(c_1), \tag{1.2}$$

where $c_t$ denotes consumption per capita at date $t$ ($t$ = 0,1). Recall that the period utility function $u(\cdot)$ is assumed to exhibit positive and diminishing marginal utility for all positive levels of consumption (i.e. $u'(c) > 0$ and $u''(c) < 0$ for all $c > 0$); and that discounting the future utility level by $\delta, 0 < \delta < 1$, captures impatience, with $\delta$ being now the subjective discount factor of the representative agent. The marginal rate of substitution associated with the utility function (1.2) is given by

$$MRS_{01}(c_0, c_1) := \frac{\partial U / \partial c_0}{\partial U / \partial c_1} = \frac{u'(c_0)}{\delta u'(c_1)}. \tag{7.1}$$

Consider first the case of a pure exchange economy. By market clearing, the per-capita net demand for bonds must by zero and hence the one-date constraints (1.24) and (1.25) yield

$$c_t = \omega_t \qquad (\mathrm{t} = 0,1) \tag{7.2}$$

in equilibrium, where $\omega_t$ is endowed real income per capita at date $t$. Clearly, the endowed income of the representative agent must be presumed to be positive at every date for equilibrium to exist. It follows from (7.2) that the first-order condition (2.3) must be satisfied at the endowment point in equilibrium; so, by using (7.1), it yields the equilibrium interest rate right away:

$$1 + r = \frac{u'(\omega_0)}{\delta u'(\omega_1)}. \tag{7.3}$$

To keep the formulas simpler, we will typically solve for $1 + r$, the gross return on bonds, rather than for the net return $r$. By comparing (7.3) to (2.19), we see that the economy's equilibrium interest rate coincides with the autarky interest rate of the representative agent.

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[46] Our exposition here draws upon Armitage (2005), Cochrane (2001), and Lengwiller (2004). For a critique of the representative-agent approach see Kirman (1992).
[47] Alternatively, individual variables may be interpreted as aggregate or economy-wide quantities.

We can see immediately from (7.3) that the equilibrium real interest rate decreases with a higher $\delta$ and rises with a higher $\omega_1$ or lower $\omega_0$. Indeed, by diminishing marginal utility, the denominator in (7.3) gets smaller if $\omega_1$ rises; or the numerator in (7.3) gets larger if $\omega_0$ falls. It is not clear though what is the impact of a change in the curvature of $u(\cdot)$, reflecting the society's aversion for intertemporal substitution (or its penchant for consumption smoothing).

To capture this additional effect on the interest rate (and for many other purposes, especially in decision problems under uncertainty), we often use a power utility form[48],

$$u(c) := \frac{1}{1-\gamma} c^{1-\gamma} \quad (\gamma > 0, \gamma \neq 1). \tag{7.5}$$

There are two important elasticity concepts associated with (7.5). Recall the concept of elasticity first[49]: if $x$ and $y$ are two positive variables, then *the elasticity of y with respect to x* is

$$\mathrm{El}_x y := \frac{dy / y}{dx / x} = \frac{d \ln y}{d \ln x}. \tag{7.6}$$

For all $c > 0$ (7.5) yields

$$u'(c) = c^{-\gamma} \tag{7.7}$$

and

$$u''(c) = -\gamma c^{-\gamma-1}, \tag{7.8}$$

establishing that (7.5) exhibits positive and diminishing marginal utility for all positive levels of consumption. From (7.7) we have $\ln u'(c) = -\gamma \ln c$, entailing a constant **elasticity of marginal utility**:

$$\mathrm{El}_c u'(c) = -\gamma. \tag{7.9}$$

Thus the larger the parameter $\gamma$, the greater is the reduction in the slope (marginal utility) of the graph of (7.5) with an increase in *c*, i.e. the graph of (7.5) gets curvier with a higher value of $\gamma$. The same holds true with the curvature of the indifference curves in the $c_0 c_1$ plane: a typical indifference curve becomes curvier with a larger $\gamma$. We may see this with the aid of a second elasticity concept—the **elasticity of inertemporal substitution**, defined as the elasticity of

48 The logarithmic utility function is a special case of (7.5) in the sense that the limit of $u(c) := \frac{1}{1-\gamma}(c^{1-\gamma} - 1)$ as $\gamma \to 1$ turns out to be $u(c) := \ln(c)$. See, for example, Eeckhoudt et al. (2005), Section 1.7

49 Sydsaeter and Hammond (2006), Section 7.7.

consumption growth $c_1 / c_0$ with respect to $MRS_{01}(c_0, c_1)$. Since an optimizing agent equates her marginal rate of substitution to one plus the real interest rate, the elasticity of intertemporal substitution measures the responsiveness of the optimal consumption ratio (consumption growth) $c_1 / c_0$ to changes in the interest rate. From (7.1) and (7.7) we obtain

$$MRS_{01}(c_0, c_1) = \frac{u'(c_0)}{\delta u'(c_1)} = \frac{1}{\delta}\left(\frac{c_1}{c_0}\right)^{\gamma}. \tag{7.10}$$

It follows that $\ln MRS_{01}(c_0, c_1) = \ln\frac{1}{\delta} + \gamma \ln\left(\frac{c_1}{c_0}\right)$ or $d \ln MRS_{01}(c_0, c_1) = \gamma d \ln\left(\frac{c_1}{c_0}\right)$ and hence

$$\mathrm{El}_{MRS_{01}}\left(\frac{c_1}{c_0}\right) = \frac{1}{\gamma}. \tag{7.11}$$

Thus a 1% increase in $MRS_{01}$ would induce a $(1/\gamma)\%$ increase in $c_1 / c_0$ (at the margin) and, therefore, the responsiveness of $c_1 / c_0$ to a 1% increase in the slope of the indifference curve declines with a greater $\gamma$. Equivalentl y, to attain a given percentage increase in $c_1 / c_0$ the required percentage augmentation in the slope of the indifference curve rises with a larger $\gamma$ (smaller elasticity of intertemporal substitution). In other words, the curvier the graph of the power utility function, the greater is the curvature of the associated indifference curves.

The best way to see the economic significance of the curvature of the indifference curves [as measured by (7.11)] is to recall the Hicks substitution effect: the agent's response to an increase in the interest rate, holding utility constant; the agent would lower current consumption and increase the amount of future consumption by rolling the lifetime budget line around the indifference curve through the original consumption stream. Clearly, the increase in the interest rate that would induce the agent to a given consumption growth is high when the curvature of the indifference curve is high. Thus a large $\gamma$ signifies a high degree of aversion to intertemporal substitution (or a strong penchant for consumption smoothing). In the extreme case when $\gamma$ gets very close to zero, the elasticity of intertemporal substitution is then very large: indifference curves are almost straight lines, so consumption now and consumption later are close to being perfect substitutes. Thus a very small $\gamma$ would portray a very low degree of aversion to unequal consumption over time. At the other extreme, i.e. when $\gamma$ becomes very large, the elasticity of intertemporal substitution gets close to zero—current and future consumption are almost perfect complements (indifference curves are nearly L-shaped), so substitutability is hardly possible. In other words, the agent's aversion to rearranging consumption over time gets extremely high.

Return to the equilibrium condition (7.3) and use (7.10) to obtain the equilibrium real interest rate as a function of $\delta, \omega_1 / \omega_0$, and $\gamma$ :

$$1 + r = \frac{1}{\delta}\left(\frac{\omega_1}{\omega_0}\right)^{\gamma}. \tag{7.12}$$

Equation (7.12) makes transparent **three** effects on the equilibrium interest rate [two of which we have already noticed in the general context of (7.3)]:

1. The equilibrium real interest rate increases with a lower subjective discount rate. In other words, the real interest rate is high when people are very impatient, i.e. when $\delta$ is low. If everybody has a very high marginal valuation of current consumption, it takes a high interest rate to convince them to save.

2. The equilibrium real interest rate increases with a higher endowed output growth (i.e. with higher $\omega_1 / \omega_0$)). So, on account of (7.2), real interest rates are high when consumption growth is high. This reflects the fact that the real saving of the agents increases with higher real interest rates.

3. Real interest rates are more sensitive to endowed output growth (and consumption growth) if the aversion to intertemporal substitution is large. Indeed, by (7.10), $\mathrm{El}_{\omega_1/\omega_0}(1+r) = \gamma$. That is, a 1% marginal increase in $\omega_1 / \omega_0$ would induce a $\gamma$% increase in the equilibrium $1+r$. As already explained, this is so because, with a larger $\gamma$, we would need a greater interest rate change to induce a given level of consumption growth.

Examine next the equilibrium real interest rate and its sensitivity to parametric changes in the full model of a two-date financial economy with production. The one-date constraints (3.3) and (3.4) and the first-order conditions (3.12) and (3.13) for individual optimization now pertain to the representative agent. By market clearing, the per-capita net demand for bonds must by zero and hence (3.3) and (3.4) yield

$$c_0 = \omega_0 - k, \tag{7.13}$$

$$c_1 = \omega_1 + y_1 \tag{7.14}$$

in equilibrium, instead of the equations in (7.2). It is worth repeating that $k$ and $y_1$ here denote physical capital and output per capita, respectively [and, of course, $c_t$ is consumption per capita at date $t(t = 0,1)$]. The production function of the representative agent is the per-capita reinterpretation of (3.1)[50]:

$$y_1 = f(k). \tag{7.15}$$

In view of (7.10), we rewrite the first-order conditions (3.3) and (3.4) as

[50] The production function here is, in effect, the per-capita version of the economy's aggregate production function. See, for example, Barro (2008), pp. 46-49.

$$1 + r = \frac{1}{\delta}\left(\frac{c_1}{c_0}\right)^{\gamma}, \tag{7.16}$$

$$1 + r = f'(k). \tag{7.17}$$

Equation (7.17) is identical with the optimal investment rule (3.13) except it is here more convenient to write $1 + r$ on the left side of the equation.

In view (7.13)-(7.15) and (7.17), (7.16) yields a single equation in $k$:

$$f'(k) = \frac{1}{\delta}\left(\frac{\omega_1 + f(k)}{\omega_0 - k}\right)^{\gamma} \qquad (0 \le k < \omega_0). \tag{7.18}$$

In light of the assumption that the production function is neoclassical, it is not hard to ascertain that equation (7.18) has a unique solution, which we shall designate by $\hat{k}$. See Fig. 7.1. The $(1+r)$-curve[51], which is the graph of the left side of equation (7.18), slopes downward on account of diminishing marginal product. The $MRS_{01}$-curve, which is the graph of the right side of equation (7.18), is upward sloping because $f(\cdot)$ is strictly increasing (by positive marginal productivity) and the denominator gets smaller as $k$ increases toward $\omega_0$. With the equilibrium investment per capita $\hat{k}$ specified, the equilibrium real interest rate $\hat{r}$ is determined by (7.17):

$$1 + \hat{r} = f'(\hat{k}). \tag{7.19}$$

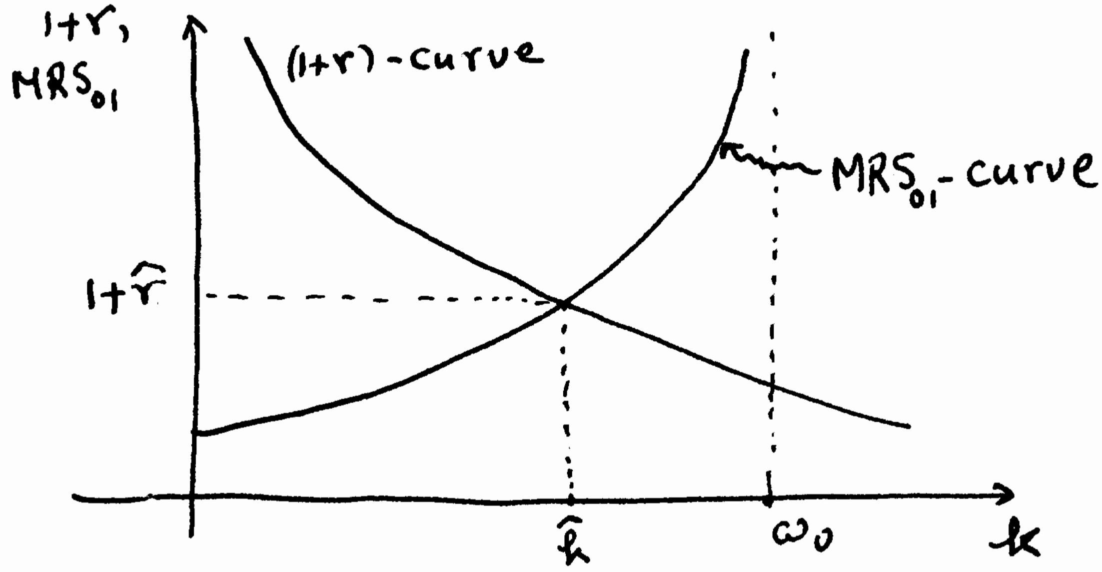


Fig. 7.1 Equilibrium Real Investment and Interest Rate

[51] This is an alternative representation of the inverse real investment demand curve. We use a different name because the canonical version of the latter has the form (3.15).

We examine first the impact of changes in impatience and endowed output stream, holding technology (the production function) constant:

• The real interest rate is high and the per-capita investment in physical capital is low when people are very impatient, i.e. when $\delta$ is low. Indeed, a decrease in the subjective discount rate will shift up the $MRS_{01}$-curve without affecting the $(1+r)$-curve: hence the equilibrium real investment will decline and the interest rate will go up.

• The real interest rate is high and the per-capita investment in physical capital is low when a large future harvest is anticipated (i.e. when $\omega_1$ is high) or when the current harvest is small (i.e. when $\omega_0$ is low). Once again, an increase in $\omega_1$ or a decrease in $\omega_0$ will shift up the $MRS_{01}$-curve without affecting the $(1+r)$-curve: hence the equilibrium real investment will decline and the interest rate will go up.

It should be noted that the above conclusions are in broad agreement with those of the pure exchange model. Whenever parametric changes do not shift the $(1+r)$-curve, the equilibrium values $\hat{k}$ and $\hat{r}$ are inversely related [by (7.19)]: if we establish that $\hat{k}$ moves in a particular direction, then we may infer that $\hat{r}$ changes in the opposite direction. This deduction is invalid if the parametric change in discussion shifts the $(1+r)$-curve as well. The $(1+r)$-curve does shift in the case of **technological advancement**: output per worker and the marginal product of capital get higher at any given level of capital per person. That is, a technological advancement amounts to an upward shift of the graph of the production function **and** the marginal product curve in Fig. 3.1. This is the case, for instance, if the production function is of the form

$$y_1 = A\sqrt{k}, \qquad (7.20)$$

where we use the symbol $A$ (a positive parameter) to represent the **technology level**[52]. The impact of technological change is as follows:

• The real interest rate is high when the technology level is high, but the connection between real investment and technological advancement is not determinate. Indeed, an increase in the technology level will shift up the $(1+r)$-curve and pivot the $MRS_{01}$-curve counterclockwise around its vertical intercept in Fig. 7.1; hence the interest rate will definitely go up but the change in real investment is ambiguous.

It would be useful to corroborate our analytically-derived conclusions by geometrical means, namely with the aid of indifference-curve/PPF diagrams[53]. In equilibrium, the per-capita net demand for bonds is zero, so the production and consumption optima of the representative individual must coincide. Hence Fig. 5.3 provides the pertinent diagram (with the suggested reinterpretation of the individual variables as quantities per capita). The economy's equilibrium

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[52] You may rewrite the production function (7.15) in the general form $y_1 = Af(k)$, and consider (7.15) as a special case with the technology level fixed at $A = 1$.

[53] See, for example, Hirshleifer and Hirshleifer (1998), Section 14.7, for a preliminary discussion.

interest rate $\hat{r}$ is the autarky or no-borrowing, no-lending interest rate $\bar{r}$ of its representative individual. Fig. 5.2 would be appropriate for the representative individual of a "small open economy", i.e. an economy which takes the *world interest rate r* as given.

Returning to the close economy equilibrium in Fig 5.3, we may examine the impact of , say a change in the degree of impatience by ascertaining whether the indifference curve becomes steeper or flatter at the original tangency point and by figuring out in turn the location of the new optimum (and equilibrium point) on the PPF. In a similar fashion, we may examine the impact of other changes: a change in the endowment will change the PPF; a technological advancement would pivot the PPF clockwise around the endowment point. The study of the impact of a technological advancement on the real interest rate is perhaps the most challenging. The trick is to carefully identify the substitution and wealth effects of the proportional upward shift of the PPF (entailed by the rise in the technology level).

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