Set 6 – Solutions

(1) $\vec{E} = -q/4\pi\varepsilon_0 r^2 \;\Theta(vt-r)\,\hat{r}$

$$\vec{\nabla}\cdot\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left[\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\Theta(vt-r) + \frac{\partial}{\partial r}\Theta(vt-r)\;\hat{r}\cdot\frac{\hat{r}}{r^2}\right]$$

Now $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(-\frac{1}{r}\right) = -\nabla^2\left(\frac{1}{r}\right) = +4\pi\,\delta(\vec{r})$

and $\partial/\partial r\;\Theta(vt-r) = -\,\delta(vt-r)$

$$\Rightarrow \boxed{\rho = \varepsilon_0\,\vec{\nabla}\cdot\vec{E} = -q\,\delta(\vec{r})\,\Theta(t) + q/4\pi r^2\;\delta(vt-r)}$$

→ negative pt. charge at the origin surrounded by an expanding shell of positive charge

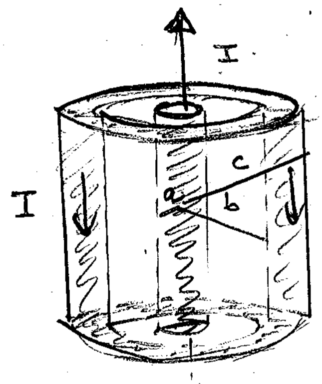
since $\vec{B}=0$, obviously $\underline{\underline{\vec{\nabla}\cdot\vec{B}=0}}$

$\vec{\nabla}\times\vec{E} = \frac{\partial E_r}{\partial\varphi}\hat{\theta} - \frac{\partial E_r}{\partial\theta}\hat{\varphi} = 0$ since $\vec{E}$ does not depend on θ, φ

$$\frac{\partial\vec{E}}{\partial t} = -\frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}\,\frac{\partial}{\partial t}\Theta(vt-r) = -\frac{qv}{4\pi\varepsilon_0 r^2}\,\delta(vt-r)\,\hat{r}$$

$$\Rightarrow \boxed{\vec{J} = -\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{qv}{4\pi r^2}\,\delta(vt-r)\,\hat{r}}$$ [since $\vec{\nabla}\times\vec{B} = 0$]

② ⓐ



Ampere's law for cylindrical symmetry

$\rightarrow \vec{B} = \mu_0 I_{int}/2\pi\rho\,\hat{\phi}$

$\underline{\rho \le a} \rightarrow I_{int} = I\frac{\rho^2}{a^2} \rightarrow \boxed{\vec{B} = \frac{\mu_0 I\rho}{2\pi a^2}\hat{\phi}}$

$\underline{a \le \rho \le b} \rightarrow I_{int} = I \rightarrow \boxed{\vec{B} = \mu_0 I/2\pi\rho\,\hat{\phi}}$

$\underline{b \le \rho \le c} \rightarrow I_{int} = I - I\frac{\rho^2 - b^2}{c^2 - b^2} \rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\frac{c^2-\rho^2}{c^2-b^2}\hat{\phi}}$

Note that $\vec{B} = 0$ for $\rho \ge c$

ⓑ $\underline{\vec{E}_{wire} = I/\pi a^2\sigma\,\hat{z}}$, $\underline{\vec{E}_{shell} = -I/\pi(c^2-b^2)\sigma\,\hat{z}}$

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \Rightarrow \boxed{\vec{S}_{wire} = -\frac{I^2\rho}{2\pi^2 a^4\sigma}\hat{\rho}}$ $(\hat{z}\times\hat{\phi} = -\hat{\rho})$

and $\boxed{\vec{S}_{shell} = +\frac{I^2(c^2-\rho^2)}{2\pi^2\rho(c^2-b^2)^2\sigma}\hat{\rho}}$

ⓒ At the surface of the wire, $\rho = a \rightarrow \rho/a^4 = 1/a^3$

$\Rightarrow \boxed{P_{wire} = 2\pi a\,\vec{S}_{wire}\cdot(-\hat{\rho}) = \frac{I^2}{\pi a^2\sigma}}$ (per unit length)

At the $\rho = c$ surface of the shell, $\vec{S}_{shell} = 0$

At the $\rho = b$ surface of the shell, $(c^2-\rho^2)/(c^2-b^2)^2 = 1/(c^2-b^2)$

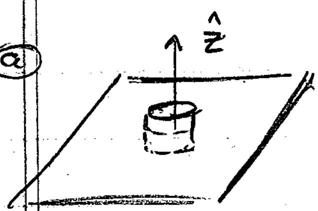
$\Rightarrow \boxed{P_{shell} = 2\pi b\,\vec{S}_{shell}\cdot(+\hat{\rho}) = \frac{I^2}{\pi(c^2-b^2)\sigma}}$ (per unit length)

ⓓ Resistance per unit length

$R = R_{wire} + R_{shell} = \frac{1}{\pi a^2\sigma} + \frac{1}{\pi(c^2-b^2)\sigma}$

$\rightarrow \boxed{P_{diss} = I^2 R = \frac{I^2}{\pi\sigma}\left(\frac{1}{a^2} + \frac{1}{c^2-b^2}\right) = P_{wire} + P_{shell}}$

Jackson (3) (a)
(6.11)



Let the screen lie in the xy-plane and let the wave travel in the $+z$ direction. The force exerted on the screen is given by $\vec{F} = \sum_{ij} \left[\oint T_{ij} (\hat{n} \cdot x_i) da \right] \hat{x}_j$

Consider a small pill box with parallel faces of area da above and below the screen. For $\hat{k} = \hat{z}$, there is no contribution to the pressure on the upper face (sheet is perfectly absorbing), and $\hat{n} = -\hat{z}$ on the lower face.

$\Rightarrow$ pressure $= \dfrac{dF_z}{da} = \sum_{ij} T_{ij} (-\hat{z} \cdot x_i)(\hat{z} \cdot \hat{x}_j) = -T_{zz}$

Now $T_{zz} = \varepsilon_0 E_z^2 - \frac{1}{2} \varepsilon_0 E^2 + \mu_0 H_z^2 - \frac{1}{2} \mu_0 H^2 = -\frac{1}{2}(\varepsilon_0 E^2 + \mu_0 H^2)$
for a wave travelling in the z-direction ($E_z = H_z = 0$)

$\rightarrow$ $\boxed{\text{pressure} = \frac{1}{2}(\varepsilon_0 E^2 + \mu_0 H^2) = u_E + u_M}$

(b) Let $\sigma =$ mass per unit area $\rightarrow$ acceler $a = \text{pres}/\sigma$

For transverse EM waves, $\mu_0 H^2 = \varepsilon_0 E^2 \Rightarrow H = E\sqrt{\varepsilon_0/\mu_0} = E/\mu_0 c$

$\Rightarrow |\vec{S}| = EH = E^2/\mu_0 c$

From part a, pres $= \varepsilon_0 E^2$ (since $\mu_0 H^2 = \varepsilon_0 E^2$)

$\Rightarrow a = \varepsilon_0 E^2/\sigma = \varepsilon_0 \mu_0 c |\vec{S}|/\sigma \rightarrow \underline{a = |\vec{S}|/\sigma c}$

(using $\varepsilon_0 \mu_0 = 1/c^2$)

Now $|\vec{S}| = 1.4 \times 10^3 \text{ W/m}^2$, $\sigma = 1.0 \times 10^{-3} \text{ kg/m}^2$

$c = 3.0 \times 10^8 \text{ m/s}$ $\rightarrow$ $\boxed{a = 4.7 \times 10^{-3} \text{ m/s}^2}$

Jackson (7.1) 4

$s_0 = 25, \quad s_1 = 0, \quad s_2 = 24, \quad s_3 = 7$

Now $\left.\begin{aligned} s_0 &= a_1^2 + a_2^2 \\ s_1 &= a_1^2 - a_2^2 \end{aligned}\right\} \Rightarrow \underline{a_1 = \sqrt{\frac{s_0+s_1}{2}} = \frac{5}{\sqrt{2}}}, \quad \underline{a_2 = \sqrt{\frac{s_0-s_1}{2}} = \frac{5}{\sqrt{2}}}$

and $2a_1a_2 = \sqrt{s_0^2 - s_1^2} = 25$

Also, $\left.\begin{aligned} s_2 &= 2a_1a_2 \cos(\delta_2-\delta_1) \\ s_3 &= 2a_1a_2 \sin(\delta_2-\delta_1) \end{aligned}\right\} \Rightarrow \underline{\tan(\delta_2-\delta_1) = \frac{s_3}{s_2} = \frac{7}{24}}$

$$\rightarrow \boxed{\vec{E} = \frac{5}{\sqrt{2}}\left\{\hat{x}\cos(kz-\omega t) + \hat{y}\cos\left[kz-\omega t + \tan^{-1}\left(\tfrac{7}{24}\right)\right]\right\}}$$

For equal x and y amplitudes, $|\vec{E}|$ is maximum when

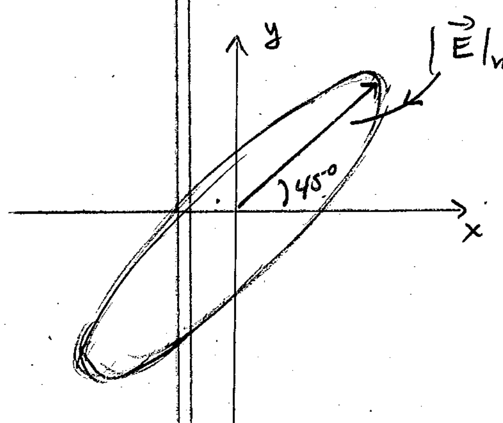
$E_y = E_x \Rightarrow \cos(kz-\omega t) = \cos(kz-\omega t+\delta) \quad [\delta \equiv \tan^{-1}(7/24)]$

$\Rightarrow \cos(kz-\omega t) = \cos(kz-\omega t)\cos\delta - \sin(kz-\omega t)\sin\delta$

$\Rightarrow \tan(kz-\omega t) = -(1-\cos\delta)/\sin\delta = -0.1429$

$\Rightarrow \cos(kz-\omega t) = \cos(kz-\omega t+\delta) = 0.9899$

$\Rightarrow \underline{|\vec{E}| = 5(0.9899) = 4.95 \text{ at maximum value}}$



In the <u>circular basis</u>

$\left.\begin{aligned} s_0 &= a_+^2 + a_-^2 \\ s_3 &= a_+^2 - a_-^2 \end{aligned}\right\} \Rightarrow \underline{a_+ = \sqrt{\frac{s_0+s_3}{2}} = 4}$

$\underline{a_- = \sqrt{(s_0-s_3)/2} = 3}$

Also, $\left.\begin{aligned} s_1 &= 2a_+a_- \cos(\delta_- - \delta_+) \\ s_2 &= 2a_+a_- \sin(\delta_- - \delta_+) \end{aligned}\right\} \Rightarrow \tan(\delta_- - \delta_+) = \frac{s_2}{s_1} = \infty$

$\Rightarrow \underline{(\delta_- - \delta_+) = \pi/2}$

$$\Rightarrow \boxed{\vec{E} = 4\hat{\varepsilon}_+\cos(kz-\omega t) - 3\hat{\varepsilon}_-\sin(kz-\omega t)}$$

(5) (a) Both the incident and transmitted electric fields have s and p polariz. components

$$\rightarrow \quad \vec{E}_I = E_I \sin\phi_I\,\hat{s} + E_I \cos\phi_I\,\hat{p}$$

$$\vec{E}_T = E_T \sin\phi_T\,\hat{s} + E_T \cos\phi_T\,\hat{s}$$

Now $\dfrac{E_{Ts}}{E_{Is}} = \dfrac{E_T \sin\phi_T}{E_I \sin\phi_I} = \dfrac{2}{1+\alpha\beta}$, $\dfrac{E_{Tp}}{E_{Ip}} = \dfrac{E_T \cos\phi_T}{E_I \cos\phi_I} = \dfrac{2}{\alpha+\beta}$

where $\alpha \equiv \cos\theta_T / \cos\theta_I \qquad \beta \equiv n_2/n_1$

Divide $\rightarrow \dfrac{\tan\phi_T}{\tan\phi_I} = \dfrac{\alpha+\beta}{1+\alpha\beta} \Rightarrow \boxed{\tan\phi_T = \left(\dfrac{\alpha+\beta}{1+\alpha\beta}\right)\tan\phi_I}$

(b) $$E_T^2 = (E_T \sin\phi_T)^2 + (E_T \cos\phi_T)^2$$

$$= \left(\frac{2}{1+\alpha\beta}\right)^2 (E_I \sin\phi_I)^2 + \left(\frac{2}{\alpha+\beta}\right)^2 (E_I \cos\phi_I)^2$$

$$= 4E_I^2 \left[\frac{\sin^2\phi_I}{(1+\alpha\beta)^2} + \frac{\cos^2\phi_I}{(\alpha+\beta)^2}\right]$$

$$\Rightarrow \boxed{E_T = 2E_I \sqrt{\frac{\sin^2\phi_I}{(1+\alpha\beta)^2} + \frac{\cos^2\phi_I}{(\alpha+\beta)^2}}}$$

(c) The transmission coefficient is given by

$$T = \alpha\beta \left|\frac{E_T}{E_I}\right|^2 = 4\alpha\beta \left[\frac{\sin^2\phi_I}{(1+\alpha\beta)^2} + \frac{\cos^2\phi_I}{(\alpha+\beta)^2}\right]$$

If $\beta = \alpha$, this reduces to

$$T = \frac{4\alpha^2 \sin^2\phi_I + (1+\alpha^2)^2 \cos^2\phi_I}{(1+\alpha^2)^2}$$

$$\rightarrow \boxed{T = \frac{4\alpha^2 + (1-\alpha^2)^2 \cos^2\phi_I}{(1+\alpha^2)^2}} \qquad (\text{using } \sin^2\phi_I = 1 - \cos^2\phi_I)$$

This reduces to

$$T_s = \frac{4\alpha^2}{(1+\alpha^2)^2} \text{ for } \cos\phi_I = 0 \qquad T_p = 1 \text{ for } \cos\theta_I = 1$$

Set 6

[Jackson 7.4] (6) (a) choose coordinates so that incident wave vector is along $\hat{z}$

Then in vacuum, $k = k_0 = \omega/c$

in metal, $k = k_R + i k_I$ where

$$\boxed{k_R = k_0 \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1+\alpha^2} + 1\right]^{1/2}} \qquad \boxed{k_I = k_0 \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1+\alpha^2} - 1\right]^{1/2}}$$

with $\boxed{\alpha = \sigma/\omega\varepsilon}$

For a good conductor, $\alpha \gg 1$

$$\Rightarrow k_R \cong k_I \cong k_0 \sqrt{\frac{\mu\varepsilon\alpha}{2}} = \frac{\omega}{c}\sqrt{\frac{2\pi\sigma\mu}{\omega}} = \frac{1}{\delta}$$

Incident wave: $\vec{E}_1 = \vec{E}_I e^{ik_0 z - i\omega t}$

Reflected wave: $\vec{E}_2 = \vec{E}_R e^{-ik_0 z - i\omega t}$

Transmitted wave: $\vec{E}_3 = \vec{E}_T e^{ikz - i\omega t}$

B.C.

$$\left.\begin{aligned} E_I + E_R &= E_T \\ k_0(E_I - E_R) &= k E_T \end{aligned}\right\} \Rightarrow E_R = \frac{k_0 - k}{k_0 + k} E_I$$

$$\Rightarrow \boxed{\left|\frac{E_R}{E_I}\right| = \left[\frac{(k_0 - k_R)^2 + k_I^2}{(k_0 + k_R)^2 + k_I^2}\right]^{1/2}} \qquad \boxed{\tan\phi = \frac{2k_0 k_I}{k_I^2 + k_R^2 - k_0^2}}$$

with expressions for wave numbers given above

(b) Reflection coefficient given by

$$R = \left|\frac{E_R}{E_I}\right|^2 = \frac{(k_0 - k_R)^2 + k_I^2}{(k_0 + k_R)^2 + k_I^2} = 1 - \frac{4k_0 k_R}{(k_0 + k_R)^2 + k_I^2}$$

For a good conductor, $k_R \cong k_I \cong 1/\delta \gg k_0 = \omega/c$

$$\Rightarrow R \cong 1 - \frac{4(\omega/c)}{\delta}\left(\frac{\delta^2}{2}\right) \rightarrow \boxed{R \cong 1 - 2\frac{\omega}{c}\delta}$$

Note also in this limit that $\tan\phi \cong \omega\delta/c$

$$\rightarrow \boxed{\phi \cong \frac{\omega}{c}\delta}$$ since ϕ small