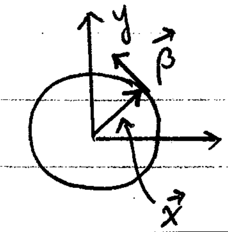
Homework Set 6 — Solutions —

(1) (a)



$\phi = \omega t \Rightarrow \boxed{\vec{x} = a[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)]}$

$\boxed{\vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt} = \frac{\omega a}{c}[-\hat{x}\sin(\omega t) + \hat{y}\cos(\omega t)]}$

$\boxed{\vec{a} = -d^2\vec{x}/dt^2 = -\omega^2\vec{x}}$

(b) $\vec{E} = \frac{q(1-\beta^2)(\vec{R}-R\vec{\beta})}{(R-\vec{R}\cdot\vec{\beta})^3} + \frac{q}{c^2}\frac{\vec{R}\times[(\vec{R}-R\vec{\beta})\times\vec{a}]}{(R-\vec{R}\cdot\vec{\beta})^3}$

Here $\vec{R} = -\vec{x} \Rightarrow \vec{R}\cdot\vec{\beta} = -\vec{x}\cdot\vec{\beta} = 0$ and $R = a$

Also $\underline{(\vec{R}-R\vec{\beta})\times\vec{a}} = \omega^2(\vec{R}-R\vec{\beta})\times\vec{R} = -\omega^2 a(\vec{\beta}\times\vec{R})$

$= +\omega^2 a(\vec{\beta}\times\vec{x}) = \underline{-(\omega a)^3\hat{z}/c}$

$\Rightarrow \vec{R}\times[(\vec{R}-R\vec{\beta})\times\vec{a}] = (\omega^3 a^4/c)[\hat{x}\sin(\omega t_r) - \hat{y}\cos(\omega t_r)]$

$1-\beta^2 = 1-(\omega a/c)^2$

$\vec{R}-R\vec{\beta} = -a\{[\cos(\omega t_r) - \omega a/c\,\sin(\omega t_r)]\hat{x} + [\sin(\omega t_r) + \omega a/c\,\cos(\omega t_r)]\hat{y}\}$

$\Rightarrow \vec{E} = \frac{q}{a^2}\left[\left(\frac{\omega a}{c}\right)^2 - 1\right]\left\{[\cos(\omega t_r) - \frac{\omega a}{c}\sin(\omega t_r)]\hat{x} + [\sin(\omega t_r) + \frac{\omega a}{c}\cos(\omega t_r)]\hat{y}\right\}$

$+ \frac{q\omega^3 a}{c^3}[\hat{x}\sin(\omega t_r) - \hat{y}\cos(\omega t_r)]$

$\rightarrow \boxed{\vec{E} = [A\cos(\omega t_r) + B\sin(\omega t_r)]\hat{x} + [A\sin(\omega t_r) - B\cos(\omega t_r)]\hat{y}}$

with $\boxed{A = \frac{q}{a^2}\left[\left(\frac{\omega a}{c}\right)^2 - 1\right]}$ and $\boxed{B = \frac{q\omega}{ac}}$

(c) $\vec{B} = \hat{R}\times\vec{E} = -(\vec{x}\times\vec{E})/a = -\hat{x}\times\vec{E}\cos(\omega t_r) - \hat{y}\times\vec{E}\sin(\omega t_r)$

$= \{[A\cos(\omega t_r) + B\sin(\omega t_r)]\sin(\omega t_r) - [A\sin(\omega t_r) - B\cos(\omega t_r)]\cos(\omega t_r)\}\hat{z}$

$= B\hat{z} \Rightarrow \boxed{\vec{B} = (q\omega/ac)\hat{z}}$

At the center of a circular loop of current, the magnetic field is (in Gaussian units)

$\vec{B} = \frac{2\pi I}{ca}\hat{z}$

Choosing $\underline{I = \omega q/2\pi \text{ gives the result above}}$

[Jackson 14.4]

2 (a) In the non-relativistic limit

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\hat{R}\times(\hat{R}\times\vec{a})\right|^2 = \frac{e^2}{4\pi c^3}\left|(\vec{a}\cdot\hat{R})\hat{R}-\vec{a}\right|^2$$

Approximate $\hat{R}\simeq\hat{r}$, then for $\vec{a}$ along $\hat{z}$, $\vec{a}\cdot\hat{R}\simeq|\vec{a}|\cos\theta$

Also for $z = a\cos(\omega_0 t)$, $|\vec{a}| = -\omega_0^2 z$

$$\Rightarrow \left|(\vec{a}\cdot\hat{R})\hat{R}-\vec{a}\right|^2 \cong |\vec{a}|^2\sin^2\theta = \omega_0^4 z^2\sin^2\theta$$

Time average of $z \rightarrow \langle z^2\rangle = \frac{1}{2}a^2$

$$\boxed{\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2\omega_0^4 a^2}{8\pi c^3}\sin^2\theta} \rightarrow \boxed{\langle P\rangle = 2\pi\int_{-1}^{1}\frac{dP}{d\Omega}\,d(\cos\theta) = \frac{e^2\omega_0^4 a^2}{3c^3}}$$

(b) If we choose the position vector along the x-axis at time $t=0$, then at time t, $\vec{x} = R(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t)$

$$\Rightarrow \vec{a} = -\omega_0^2\vec{x}$$

If we again approximate $\hat{R}\cong\hat{r}$ in the radiation zone, then

$$\vec{a}\cdot\hat{R} \cong -\omega_0^2 R(\hat{x}\cdot\hat{r}\cos\omega_0 t + \hat{y}\cdot\hat{r}\sin\omega_0 t)$$

$$= -\omega_0^2 R\sin\theta(\cos\phi\cos\omega_0 t + \sin\phi\sin\omega_0 t) = -\omega_0^2 R\sin\theta\cos(\phi-\omega_0 t)$$

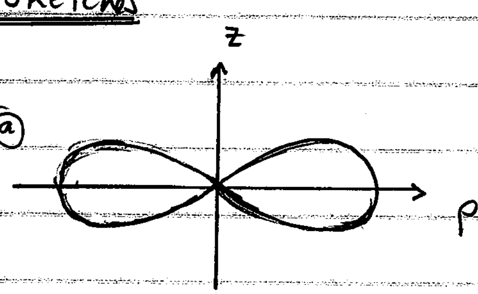
$$\Rightarrow\left|(\vec{a}\cdot\hat{R})\hat{R}-\vec{a}\right|^2 = a^2-(\vec{a}\cdot\hat{R})^2 = \omega_0^4 R^2\left[1-\sin^2\theta\cos^2(\phi-\omega_0 t)\right]$$

Time average of $\cos^2(\phi-\omega t) \rightarrow \frac{1}{2}$

$$\Rightarrow\boxed{\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2\omega_0^4 R^2}{4\pi c^3}\left(1-\frac{1}{2}\sin^2\theta\right)} \rightarrow \boxed{\langle P\rangle = \frac{2}{3}\frac{e^2\omega_0^4 R^2}{c^3}}$$

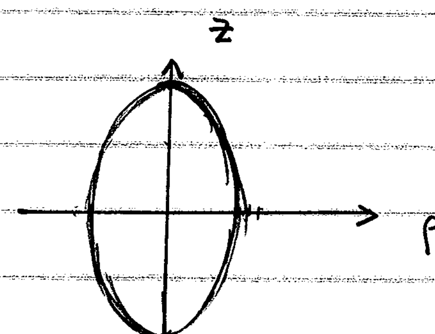
Sketches

(a)



radiation emitted $\perp$ to motion

(b)



$dP/d\Omega\,(\theta=0) = 2\,dP/d\Omega\,(\theta=\pi/2)$

[Jackson 14.5] ③ ⓐ Non relativistically, for a head-on collision with a central force field,

$$a = \frac{F}{m} = -\frac{1}{m}\frac{dV}{dr} \Rightarrow P = \frac{2}{3}\frac{(ze)^2 a^2}{c^3} = \frac{2}{3}\left(\frac{ze}{m}\right)^2 \frac{1}{c^3}\left|\frac{dV}{dr}\right|^2$$

Define $\boxed{\alpha \equiv z^2 e^2 / m^2 c^3}$

$$\rightarrow \Delta W_{in} = \int_0^{t_{min}} P\,dt = \frac{2}{3}\alpha \int_0^{t_0} \left|\frac{dV}{dr}\right|^2 dt \quad \text{with } r = r_{min} \text{ at } t = t_{min}$$

Now energy conservation $\Rightarrow \frac{1}{2} m v^2 + V = V(r_{min})$
(since $v = 0$ at $r = r_{min}$ for a head-on collision)

$$\Rightarrow \frac{dr}{dt} = -v = -\sqrt{\frac{2}{m}\left[V(r_{min}) - V\right]} \quad \left(\text{note that } \frac{dr}{dt} < 0\right)$$

$$\Rightarrow \Delta W_{in} = \frac{2}{3}\alpha \int_\infty^{r_{min}} \left|\frac{dV}{dr}\right|^2 \left(\frac{dt}{dr}\right) dr = \frac{2}{3}\alpha\sqrt{\frac{m}{2}} \int_{r_{min}}^\infty \left|\frac{dV}{dr}\right|^2 \frac{dr}{\sqrt{V(r_{min}) - V}}$$

Now the same energy is radiated as the particle moves back out to $r = \infty$

$$\Rightarrow \boxed{\Delta W = 2\Delta W_{in} = \frac{4}{3}\frac{z^2 e^2}{m^2 c^3}\sqrt{\frac{m}{2}} \int_{r_{min}}^\infty \left|\frac{dV}{dr}\right|^2 \frac{dr}{\sqrt{V(r_{min}) - V}}}$$

ⓑ For the Coulomb potential,

$$\left|\frac{dV}{dr}\right|^2 = \left(\frac{zZe^2}{r^2}\right)^2 \quad \text{and} \quad V(r_{min}) = \frac{1}{2} m v_0^2 \Rightarrow r_{min} = \frac{2zZe^2}{m v_0^2}$$

Define $a \equiv zZe^2$

$$\Rightarrow \Delta W = \frac{4}{3}\left(\frac{ze}{m}\right)^2 \frac{1}{c^3}\sqrt{\frac{m}{2}} \int_{r_{min}}^\infty \frac{a^2}{r^4} \frac{dr}{\sqrt{a/r_{min} - a/r}}$$

Define $u \equiv a/r$ and $u_{max} = a/r_{min} = \frac{1}{2} m v_0^2$

$$\rightarrow du = -\frac{a}{r^2}dr = -\frac{1}{a}u^2 dr \Rightarrow \frac{a^2}{r^4}dr = \frac{1}{a^2}u^4 dr = -\frac{u^2}{a}du$$

(3) (b) (continued)

$$\Rightarrow \int_{r_{min}}^{\infty} \frac{a^2}{r^4} \frac{dr}{\sqrt{a/r_{min} - a/r}} = \int_0^{u_{max}} \frac{u^2}{a} \frac{du}{\sqrt{u_{max} - u}}$$

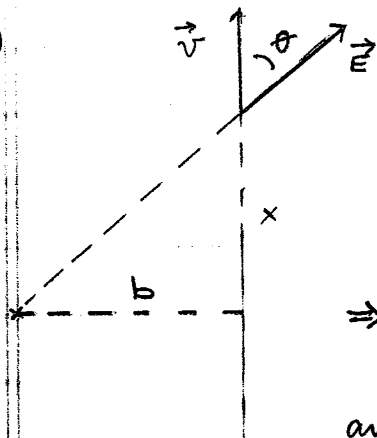
$$= -\frac{2u^2}{a}\sqrt{u_{max}-u}\,\Big|_0^{u_{max}} + \int_0^{u_{max}} \frac{4u}{a}\sqrt{u_{max}-u}\; du$$

$$= -\frac{8}{3}\frac{u}{a}(u_{max}-u)^{3/2}\Big|_0^{u_{max}} + \frac{8}{3a}\int_0^{u_{max}} (u_{max}-u)^{3/2}\, du = -\frac{16}{15}\frac{1}{a}(u_{max}-u)^{5/2}\Big|_0^{u_{max}}$$

$$= \frac{16}{15}\frac{u_{max}^{5/2}}{a} = \frac{16}{15}\frac{1}{a}\left(\frac{1}{2}mv_0^2\right)^{5/2} = \frac{4m^2 v_0^5}{15a}\sqrt{\frac{m}{2}}$$

$$\Rightarrow \boxed{\Delta W = \frac{8}{45}\frac{z\, m}{Z c^3} v_0^5}$$

(4) [Jackson 14.8]



For a <u>relativistic</u> particle in an electric field,

$$\gamma m\vec{a} = q[\vec{E} - (\vec{E}\cdot\vec{\beta})\vec{\beta}]$$

$$\Rightarrow a^2 = \left(\frac{q}{\gamma m}\right)^2\left[E^2 + (\beta^2 - 2)(\vec{E}\cdot\vec{\beta})^2\right] \qquad (q = ze)$$

and $(\vec{\beta}\cdot\vec{a})^2 = (q/\gamma m)^2(1-\beta^2)^2(\vec{E}\cdot\vec{\beta})^2$

The relativistic Lienard formula for the power gives

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^6\left[a^2 - (\vec{\beta}\times\vec{a})^2\right] = \frac{2}{3}\frac{q^2}{c^3}\gamma^6\left[(1-\beta^2)a^2 + (\vec{\beta}\cdot\vec{a})^2\right]$$

$$= \frac{2}{3}\frac{(ze)^4}{m^2c^3}\gamma^4\left\{(1-\beta^2)\left[E^2 + (\beta^2-2)(\vec{E}\cdot\vec{\beta})^2\right] + (1-\beta^2)^2(\vec{E}\cdot\vec{\beta})^2\right\}$$

$$= \frac{2}{3}\frac{(ze)^4}{m^2c^3}\gamma^2\left[E^2 - (\vec{E}\cdot\vec{\beta})^2\right] = \frac{2}{3}\frac{(ze)^4}{m^2c^3}\gamma^2E^2(1-\beta^2\cos^2\theta)$$

(using $1-\beta^2 = \gamma^{-2}$)

Now $E = \dfrac{Ze}{r^2}$ and $\cos\theta = \dfrac{x}{r}$ with $r^2 = x^2 + b^2$

If we approx. a straight line trajectory with constant $v = dx/dt$ then the total radiated energy is

$$\Delta W = \int_{-\infty}^{\infty} P\,dt = 2\int_0^\infty P\,dt = 2\int_0^\infty P\left(\frac{dt}{dx}\right)dx = \frac{2}{\beta c}\int_0^\infty P\,dx$$

$$= \frac{4}{3}\frac{z^4Z^2e^6}{m^2c^4\beta}\gamma^2\int_0^\infty dx\left[\frac{1}{(x^2+b^2)^2} - \beta^2\frac{x^2}{(x^2+b^2)^3}\right]$$

$$\int_0^\infty\frac{dx}{(x^2+b^2)^2} = \overset{0}{\cancel{\frac{x}{2b^2(x^2+b^2)}}}\Bigg|_0^\infty + \frac{1}{2b^2}\int_0^\infty\frac{dx}{x^2+b^2} = \frac{1}{2b^3}\tan^{-1}\left(\frac{x}{b}\right)\Bigg|_0^\infty = \frac{\pi}{4b^3}$$

$$\int_0^\infty\frac{x^2\,dx}{(x^2+b^2)^3} = \overset{0}{\cancel{\frac{-x}{4(x^2+b^2)^2}}}\Bigg|_0^\infty + \frac{1}{4}\int_0^\infty\frac{dx}{(x^2+b^2)^2} = \frac{\pi}{16b^3}$$

$$\rightarrow \boxed{\Delta W = \frac{\pi z^4Z^2e^6}{4m^2c^4\beta}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3}}$$

[using $1 - \frac{1}{4}\beta^2 = \frac{1}{4}(3 + \gamma^{-2})$]

5 (a) The spectral distribution is given by

$$\frac{d^2W}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt_r\, e^{i\omega(t_r - \hat{R}\cdot\vec{x}/c)}\, \hat{R}\times(\hat{R}\times\vec{\beta})\right|^2$$

Radiation zone $\longrightarrow$ $\hat{R} \cong \hat{r}$

Non-relativistic approx $\longrightarrow$ $t_r \gg \hat{R}\cdot\vec{x}/c$

$$\rightarrow \hat{R}\times(\hat{R}\times\vec{\beta}) \cong \hat{r}\times(\hat{r}\times\vec{\beta}) = (\hat{r}\cdot\vec{\beta})\,\hat{r} - \vec{\beta} = \beta(\cos\theta\,\hat{r} - \hat{z})$$

Now $|\cos\theta\,\hat{r} - \hat{z}|^2 = \cos^2\theta + 1 - 2\cos\theta\,\hat{r}\cdot\hat{z}$

$$= 1 - \cos^2\theta = \sin^2\theta$$

Also $\beta = \beta_0 \cos(\omega_0 t)\, e^{-\alpha t}$

$$\Rightarrow \frac{d^2W}{d\omega\, d\Omega} = \frac{q^2\omega^2\beta_0^2}{4\pi^2 c}\sin^2\theta\left|\int_0^{\infty} dt_r\, e^{(i\omega-\alpha)t_r}\cos(\omega_0 t_r)\right|^2$$

$$= \left(\frac{q\omega\beta_0}{2\pi}\right)^2\frac{\sin^2\theta}{c}\left|\int_0^{\infty} e^{i(\omega-\alpha)t_r}\,\frac{1}{2}\left(e^{i\omega_0 t_r} + e^{-i\omega_0 t_r}\right)\right|^2$$

$$\int = \frac{1}{2}\left[\frac{e^{[i(\omega+\omega_0)-\alpha]t_r}}{i(\omega+\omega_0)-\alpha} + \frac{e^{[i(\omega-\omega_0)-\alpha]t_r}}{i(\omega-\omega_0)-\alpha}\right]_0^{\infty}$$

$$= -\frac{1}{2}\,\frac{[i(\omega+\omega_0)-\alpha] + [i(\omega-\omega_0)-\alpha]}{\omega_0^2-\omega^2+\alpha^2-2i\alpha\omega} = -\frac{i\omega-\alpha}{\omega_0^2-\omega^2+\alpha^2-2i\alpha\omega}$$

$$\Rightarrow \left|\int\right|^2 = \frac{\omega^2+\alpha^2}{(\omega_0^2-\omega^2+\alpha^2)^2+4\alpha^2\omega^2}$$

$$\Rightarrow \boxed{\frac{d^2W}{d\omega\, d\Omega} = \left(\frac{q\omega\beta_0}{2\pi}\right)^2\frac{\sin^2\theta}{c}\,\frac{\omega^2+\alpha^2}{(\omega_0^2-\omega^2+\alpha^2)^2+4\alpha^2\omega^2}}$$

5. For $\omega \cong \omega_0$ and $\alpha << \omega_0$, in the numerator

$$\omega^2 + \alpha^2 \cong \omega_0^2$$

In denominator,

$$(\omega_0^2 - \omega^2 + \alpha^2)^2 = \left[(\omega_0 + \omega)(\omega_0 - \omega) + \alpha^2\right]^2$$

$$\cong \left[2\omega_0(\omega_0 - \omega) + \alpha^2\right]^2 \cong 4\omega_0^2(\omega_0 - \omega)^2 + 4\omega_0(\omega_0 - \omega)\alpha^2$$

Also, $4\omega_0(\omega_0 - \omega)\alpha^2 + 4\omega^2\alpha^2 \cong 4\omega_0^2\alpha^2$

$$\Rightarrow \frac{\omega^2 + \alpha^2}{(\omega_0^2 - \omega^2 + \alpha^2)^2 + 4\alpha^2\omega^2} \cong \frac{1}{4}\,\frac{1}{(\omega_0 - \omega)^2 + \alpha^2}$$

$$\Rightarrow \boxed{\frac{d^2W}{d\omega\, d\Omega} \cong \left(\frac{q\beta_0 \sin\theta}{4\pi}\right)^2 \frac{1}{c}\,\frac{\omega^2}{(\omega_0 - \omega)^2 + \alpha^2}}$$