CHAPTER TEN

SPECIAL TOPICS IN FINANCE

Time Value of Money: The case of Arithmetic and Geometric growth and their Applications

I. Introduction

Knowledge of how interest compounds is a cornerstone of finance and is integral in financial decisions at the personal and corporate level. Albert Einstein is quoted to have replied to the question, “What is man’s greatest discovery”, “compound interest”, without hesitation. From its first introduction, several centuries ago, computing compound interest and valuations formulated in terms of constant interest rates have been facilitated by compound interest tables, computer and calculator algorithms, and simple closed form equations like the annuity or constant geometric dividend growth model by Gordon. These can be found in virtually every finance text Brigham and Ehrhardt (2002, pp 385-396), Brealy, Myers & Marcus (1999, pp 54-89). Every finance text in investments, real estate, corporate finance, etc. devote at least one or more chapters to developing present and future value compound interest rules and relationships(See for example- Besley & Brigham (2005, pp 88-125), Prakash, Karels & Fernandez (1987, pp 15-63). These developments are also found in basic economics texts and texts in engineering economics, and present value and future rules can also be traced to other disciplines like Marketing, Decision Sciences, Accounting, etc.

When the interest rate is assumed to remain constant, the present and future value of a series of cash flows that remain constant or grow at a constant rate are easily defined in terms of simple equations. Basic finance texts like Ross, Westerfield and Jordan (2004, pp 87-104), Van Horne (2002, pp 12-35), Brealey, Myers and Allen (2006, pp 35-48) provide these equations for the cases of annuities, zero growth, and constant geometric growth when the series continues indefinitely.

Unfortunately, the texts limit the development of simple PV-FV rules to these specific cash flow series and therefore do not provide the student with the capacity to value cash flows following different growth patterns like finite series of constant geometric growth or cash flows with constant arithmetic growth. Although some finance texts offer the finite geometric growth formulation, not a single text addresses the arithmetic growth case. The engineering economics texts are the only source for PV-FV rules for cash flows following arithmetic growth, and these texts limit the development more to the mathematics of sums of series rather than the development of simple rules for valuation when the cash flows are finite or multiple stages of arithmetic growth.
For example, if you had covered the topic of geometric growth in cash flows, in the valuation of common stocks under Gordon’s valuation model would have been a simple solution rather than the one which one finds in the standard textbooks such as Brigham and Ehrhardt (2002, pp.385-396), Brealy, Myers & Marcus, (1999, pp.54-89). During the course of our teaching we have found that if we use the present value formulation of future cash flows under geometric growth, the valuation of the stock under super normal growth would have taken one-tenth of the time which we usually spend in teaching the students in the classroom. On the other hand, the case of arithmetic growth in cash flow is intuitively more plausible for a variety of applications in areas such as Life cycle, Piece-wise approximation, Macaulay’s duration, Engineering economics “Constant gradient” etc.

In the sections to follow, simple closed form equations are developed for the valuations of cash flows following both geometric and arithmetic growth when the interest rate is constant. Equations for both finite cash flows and cash flows continuing to infinity are developed. The developments are presented as sums of series and for the case of finite cash flows as a difference between the PV of infinite series starting at different points in time. Intuitive developments are also introduced for those knowledgeable of PV rules for annuities and perpetuities for the case of overlooked arithmetic growth series. It is the intent of this research to show the ease with which PV-FV may be derived for the broad variety of applications of both the more widely known geometric growth and the generally omitted arithmetic growth. By consolidating the developments of both the geometric and arithmetic growth PV-FV rules in this one paper, we hope to motivate the introduction of both models into finance texts for both finite and infinite series. As a final note, a number of applications of arithmetic growth are offered.

**Section – II**

**Present value of a series of cash flow with finite and infinite geometric growth:**

Consider the following time line

\[
\begin{align*}
\text{CF} & \quad \text{CF} (1+g) \quad \text{CF} (1+g)^2 \quad \cdots \quad \text{CF} (1+g)^{n-1} \\
\end{align*}
\]

where CF is a constant amount, g is the growth rate, r is the constant discount rate, and n is the number of periods. The present value at time 0 of this cash flow, assuming a discount rate of r, is given by

\[
PV_0 = \frac{CF}{1+r} + \frac{CF(1+g)}{(1+r)^2} + \cdots + \frac{CF(1+g)^{n-1}}{(1+r)^n}
\]

\[
= \frac{CF}{1+r} \left[ \frac{1 + \frac{(1+g)}{(1+r)}}{1 + \frac{(1+g)^{n-1}}{(1+r)^n}} \right]
\]

The series under the bracket is a geometric series with common ratio \(\frac{1+g}{1+r}\). Hence,
PV = \frac{CF}{1+r} \left[ \frac{1 - \left( \frac{1 + g}{1 + r} \right)^n}{1 - \frac{1 + g}{1 + r}} \right]

PV = \frac{CF}{r - g} \left[ 1 - \left( \frac{1 + g}{1 + r} \right)^n \right] \quad \text{(10.1)}

If however \( n \rightarrow \infty \) we have

PV = \frac{CF}{r - g} \left[ 1 - \left( \frac{1 + g}{1 + r} \right)^n \right] \text{as}

PV = \frac{CF}{r - g} \quad \text{for} \ r > g \quad \text{(10.2)}

= \infty \quad \text{for} \ r \leq g \quad \text{(10.3)}

The expression (10.2) is the familiar stock valuation formula, where CF can be replaced by dividend next period \( D_1 \).

**Illustration:**

Mr. Stevens will deposit $100 in the first year. Beginning second year his deposit will increase by 10% per year for next 14 years. Assuming rate of interest of 12% compounded annually;

(a) What will be the present value of his deposit?
(b) What will be the present value of his deposits if it grows at 10% each year forever?

(a) We depict the time line of deposits as follows:

<table>
<thead>
<tr>
<th>100</th>
<th>100(1+.10)</th>
<th>100(1+.10)^2</th>
<th>100(1+.10)^{14}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>..............</td>
<td>.................</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
</tbody>
</table>

In this problem we have

\( CF = 100 \)

\( g = .10 \)

\( n = 15 \)

\( r = .12 \)

Substituting these values in expression (1) we have

\[
PV = \left[ \frac{100}{.12 -.10} \right] \left[ 1 - \left( \frac{1+.10}{1+.12} \right)^{15} \right]
\]
(b) When his deposits increases by 10% forever for \( r > g \), we apply the formula derived above (expression 2) for infinite period to find out the present value.

Thus, we have

\[
P V = \frac{CF}{r - g}
\]

\[
= \frac{100}{.12 - .10}
\]

or \( PV = $ 5000 \)

Furthermore the ways stock valuations are done in the textbook (usually the supernormal growth cases) are usually quite burdensome and tedious. Using the formula derived in this section, one can solve any problem under supernormal growth no matter how long it persists.

This geometric growth formulation of present value can easily be extended to stock valuation under supernormal growth. For definition of supernormal growth is that the dividend grows at a rate higher than the discount rate for ‘n’ number of years, then it stabilizes at a rate less than the discount rate forever. Thus if we assume \( g_s = \) supernormal growth \( (r \leq g) \) for \( n \) years, and then after the \( n^{th} \) year the rate of growth reverts to normal growth pattern of say, \( g_n < r \)

In order to value these cashflow patterns consider the following timeline:

<table>
<thead>
<tr>
<th>CF</th>
<th>CF ((1+g_s))</th>
<th>CF ((1+g_s)^2)</th>
<th>CF ((1+g_s)^{n-1})</th>
<th>CF ((1+g_s)^n)((1+g_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>n</td>
</tr>
<tr>
<td>n+1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\infty)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The present value is then

\[
PV = \frac{CF}{(1+r)} + \frac{CF(1+g_s)}{(1+r)^2} + ... + \frac{CF(1+g_s)^{n-1}}{(1+r)^n} + \frac{CF(1+g_s)^n(1+g_n)}{(1+r)^{n+1}} \left[ \frac{1 + g_n}{(1 + r)^{n+1}} + \frac{(1 + g_n)^2}{(1 + r)^{n+2}} + ... \right]
\]

\[
PV = \frac{CF}{r - g_s} \left[ 1 - \left( \frac{1 + g_s}{1 + r} \right)^n \right] + \frac{CF(1+g_s)^{n-1}(1+g_n)}{(1+r)^{n+1}} \left[ \frac{1 + g_n}{1 + r} + \left( \frac{1 + g_n}{1 + r} \right)^2 \right] + ... \]

The sum of the second expression above is
Therefore the present value of the stock under supernormal growth will be

\[
PV = \frac{CF_1 (1 + g_s)^{r-1} (1 + g_\infty)}{(1 + r)^r} \frac{1}{r - g_\infty}
\]

The above stock valuation method can easily be used to obtain the value of common stock under supernormal growth. To illustrate we consider an example from the “Financial Management: Theory and practice (2002) by Eugene F. Brigham & Michael C. Ehrhardt” (2002, pp. 395-396):

“We can use the constant growth formula, Equation 10.2 to determine what the stock’s horizon or terminal value will be N periods from today:

\[
\text{Horizon value} = \frac{D_{N+1}}{k_s - g} = \frac{D_N (1 + g)}{k_s - g}
\]

The stock’s intrinsic value today, 0, is the present value of the dividends during the non constant growth period plus the present value of the horizon value:

\[
P_0 = \frac{D_1}{(1 + k_s)^1} + \frac{D_2}{(1 + k_s)^2} + \frac{D_3}{(1 + k_s)^3} + \ldots + \frac{D_N}{(1 + k_s)^N} + \frac{D_{N+1}}{(1 + k_s)^{N+1}} + \ldots + \frac{D_\infty}{(1 + k_s)^\infty}
\]

Where

\[
\hat{P}_N = \frac{[D_{N+1}](k_s - g)}{(1 + k_s)^N}
\]

To implement Equation 10.5, we go through the following three steps:

1. Find the PV of the dividends during the period of nonconstant growth.
2. Find the price of the stock at the end of the nonconstant growth period, at which point it has become a constant growth stock, and discount this price back to the present.
3. Add these two components to find the intrinsic value of the stock, 0.

Figure 10.3 can be used to illustrate the process for valuing nonconstant growth stocks. Here we assume the following five facts exist:

\[
k_s = \text{stockholder’s required rate of return} = 13.4\%.
\]

\[
N = \text{years of supernormal growth} = 3
\]

\[
g_\infty = \text{rate of growth in both earning and dividends during the supernormal growth period} = 30\%.
\]
from year to year. Also, there could be several different supernormal growth periods, e.g., 30% for three years, then 20% for three years, and then a constant 8%.

\[ g_n = \text{rate of normal, constant growth after the supernormal period} = 8\% \]

This rate is also shown on the time line, between periods 3 and 4.

\[ D_0 = \text{last dividend the company paid} = \$1.15 \]

**Figure 10.3 (Process for finding the value of supernormal growth stock)**

<table>
<thead>
<tr>
<th>( g_s = 10% )</th>
<th>30%</th>
<th>30%</th>
<th>( g_n = 8% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 = 1.4950 )</td>
<td>( D_2 = 1.9435 )</td>
<td>( D_3 = 2.5266 )</td>
<td>( D_4 = 2.7287 )</td>
</tr>
</tbody>
</table>

\[ \frac{1.3183}{1.5113} \]

\[ \frac{36.3838}{50.5310} \]

\[ 39.2134 = \$ 39.21 = \hat{P}_0 \]

**Step 1.**

Calculate the dividends expected at the end of each year during the supernormal growth period. Calculate the first dividend, \( D_1 = D_0 (1 + g_s) = \$1.15(1.30) = \$1.4950 \). Here, \( g_s \) is the growth rate during the three year supernormal growth period, 30 percent. Show the $1.4950 on the time line as the cash flow at Time 1. Then, calculate \( D_2 = D_1 (1 + g_s) = \$1.4950(1.30) = \$1.9345 \), and then \( D_3 = D_2 (1 + g_s) = \$1.9345(1.30) = \$2.5266 \). Show these values on the time line as the cash flows at Time 2 and Time 3. Note that \( D_0 \) is used only to calculate \( D_1 \).

**Step 2.**

The price of the stock is the PV of dividends from time 1 to infinity, so in theory we could project each future dividend, with the normal growth rate, \( g_n = 8\% \), used to calculate \( D_4 \) and subsequent dividends. However, we know that after \( D_3 \) has been paid, which is at time 3, the stock becomes a constant growth stock. Therefore, we can use the constant growth formula to find \( \hat{P}_3 \), which is the PV of the dividends from time 4 to infinity as evaluated at time 3.

First, we determine \( D_4 = \$2.5266(1.08) = \$2.7287 \) for use in the formula, and then we calculate \( \hat{P}_3 \), as follows:

\[
\hat{P}_3 = \frac{D_4}{k_s - g_n} = \frac{\$2.7287}{0.134 - 0.08} = \$50.5310
\]

We show this $50.5310 on the time line as a second cashflow at time 3. The $50.5310 is a time 3 cash flow in the sense that the owner of the stock could sell it for $50.5310 at time 3 and also in the sense that $50.5310 is the present value of the dividend cash flows from time 4 to infinity. Note that the total cashflow at time 3 consists of the sum of \( D_3 + \hat{P}_3 = \$2.5266 + \$50.5310 = \$53.0576 \)

**Step 3.**

Now that the cash flows have been placed on the time line, we can discount each cash flow at the required rate of return, \( k_s = 13.4\% \). We could discount each cash flow by dividing by \( (1.134)^t \), where \( t = 1 \) for Time 1, \( t = 2 \) for Time 2, and \( t = 3 \) for time 3. This produces the PVs shown to the left below the time line, and the sum of the PVs is the value of supernormal growth stock, $39.21"
Same problem can be solved by the formula developed in section 2. Using expression (10.4), i.e.

\[
PV = \frac{CF_1}{r - g_s} \left[ 1 - \left( \frac{1 + g_s}{1 + r} \right)^n \right] + \frac{CF_1 (1 + g_s)^{n-1} (1 + g_n)}{(1 + r)^n (r - g_n)}
\]

\[
= \frac{1.495}{.134 - .30} \left[ 1 - \left( \frac{1 + .30}{1 + .134} \right)^3 \right] + \frac{1.495(1 + .30)^2(1 + .08)}{(1 + .134)^3(.134 - .08)}
\]

\[
= \$ 39.213
\]

There is another advantage of this formula. For example, suppose in the above example, the growth persists for 30 years. Using the procedure as outlined in the above textbook one can obtain the value, but it will be extremely time consuming. Using the formula derived in this section, computational difficulty is eased considerably.

**Section : III**

**Present value of cash flows with arithmetic growth:**

Consider the cash flow pattern given below:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>CF + d</td>
<td>CF + 2d</td>
<td>CF + 3d</td>
<td>CF + (n-1)d</td>
<td></td>
</tr>
</tbody>
</table>

where CF is the cash flow in period 1 and d is the constant level growth until n, the present value will be

\[
PV = \frac{CF}{(1+r)} + \frac{CF + d}{(1+r)^2} + \frac{CF + 2d}{(1+r)^3} + ... + \frac{CF + (n-1)d}{(1+r)^n}
\]

where \( r \) is the discount rate. Rewriting the above expression we have

\[
PV = \frac{CF}{(1+r)} + \frac{CF}{(1+r)^2} + \frac{CF}{(1+r)^3} + ... + \frac{CF}{(1+r)^n} + \left\{ \frac{d}{(1+r)^2} + \frac{2d}{(1+r)^3} + ... + \frac{(n-1)d}{(1+r)^n} \right\}
\]

\[
PV = PV \text{ of an annuity} + \text{Sum of a series (S)}
\]

\[
PV = CF \left[ \frac{1 - (1+r)^{-n}}{r} \right] + S
\]
Now let us evaluate $S$

\[ S = \frac{d}{(1+r)^2} + \frac{2d}{(1+r)^3} + \ldots + \frac{(n-2)d}{(1+r)^{n-1}} + \frac{(n-1)d}{(1+r)^n} \quad \text{(10.5)} \]

Multiplying both sides by $\frac{1}{(1+r)}$, we have

\[ \frac{1}{(1+r)} S = \frac{d}{(1+r)^3} + \frac{2d}{(1+r)^4} + \ldots + \frac{(n-2)d}{(1+r)^n} + \frac{(n-1)d}{(1+r)^{n+1}} \quad \text{(10.6)} \]

Subtracting (6) from (5) we have

\[ S \left[ 1 - \frac{1}{1+r} \right] = \frac{d}{(1+r)^2} + \frac{d}{(1+r)^3} + \frac{d}{(1+r)^4} + \ldots + \frac{d}{(1+r)^n} - \frac{(n-1)d}{(1+r)^{n+1}} \]

The first (n-1) terms are geometric series with common ratio $\frac{1}{1+r}$. Therefore its sum will be;

\[ S \left[ 1 - \frac{1}{1+r} \right] = \left( \frac{d}{(1+r)^2} \left( 1 - \left( \frac{1}{1+r} \right)^{n-1} \right) \right) - \left( \frac{(n-1)d}{(1+r)^{n+1}} \right) \]

\[ S \left[ \frac{r}{(1+r)} \right] = \frac{d}{(1+r)^2} \left[ 1 - (1+r)^-(n-1) \right] \left( \frac{r}{1+r} \right) - \left( \frac{(n-1)d}{(1+r)^{n+1}} \right) \]

\[ S = \frac{d}{r^2} \left[ 1 - (1+r)^-(n-1) \right] - \frac{(n-1)d}{r(1+r)^n} \]

Therefore,

\[ PV = CF \left[ \frac{1-(1+r)^{-n}}{r} \right] + \frac{d}{r^2} \left[ 1 - (1+r)^-(n-1) \right] - \frac{(n-1)d}{r(1+r)^n} \quad \text{(10.7)} \]
Section: IV

Special cases under Arithmetic Growth

1. Growing up to infinity (n → ∞)

By expression (7) above, we have

\[ PV = CF \left( \frac{1 - (1 + r)^{-n}}{r} \right) + \frac{d}{r^2} \left[ 1 - (1 + r)^{-(n-1)} \right] - \frac{(n-1)d}{r(1 + r)^n} \ldots (10.7) \]

for \( n \to \infty \) and \( r > 0 \), we have the following:

1) \( CF \left( \frac{1 - (1 + r)^{-n}}{r} \right) \to \frac{CF}{r} \ldots (10.8) \)

2) \( \frac{d}{r^2} \left[ 1 - (1 + r)^{-(n-1)} \right] \to \frac{d}{r^2} \ldots (10.9) \)

3) \( \frac{(n-1)d}{r(1 + r)^n} \to 0 \), by L’hôpital’s rule \ldots (10.10) \)

Substituting the expressions (10.8), (10.9), (10.10) in expression (10.7), we have

\[ PV = \frac{CF}{r} + \frac{d}{r^2} \ldots \ldots \ldots (10.11) \]

Illustration:

Mr. Stevens will deposit $100 at the end of first year. Beginning second year he expects to deposit extra $10 each year for next 19 years. Assuming rate of interest of 5% compounded annually

(a) What will be the present value of Mr. Stevens’s deposits today?
(b) What will be the present value if deposit increases by $10 each year for forever?

(a) We draw the time line and use the derived equation to find out the present value.

<table>
<thead>
<tr>
<th></th>
<th>$100</th>
<th>$110</th>
<th>$190</th>
<th>$200</th>
<th>$290</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
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</tbody>
</table>

Substituting the numerical values in expression (10.7), we have,
PV = \left( \frac{1-(1+.05)^{-20}}{.05} \right) + \frac{10}{(05)^2} \left[ 1-(1+.05)^{-10} \right] - \frac{19 \times 10}{.05(1+.05)^{20}}

= $ 2231.10

b) We depict the time line of deposits as follows:

<table>
<thead>
<tr>
<th></th>
<th>$100</th>
<th>$110</th>
<th>$190</th>
<th>$200</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
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<td>2</td>
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<td>10</td>
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<td></td>
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<td>11</td>
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<tr>
<td>………..\infty</td>
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</tr>
</tbody>
</table>

Substituting the numerical values in expression (10.11), we have

PV = \frac{100}{.05} + \frac{10}{(05)^2}

or PV = $ 6000

2. **Alternative derivation of the present value under arithmetic growth**

Assume that the cash flows are growing by an amount \( d \) each period until infinity. If we subtract from this expression another cash-flow series starting at \( n \) period hence and extending to infinity, the resultant series will be the present value of a finite arithmetic growth series. That is,

\[
PV_0 = PV_0^{1+\infty} - PV_0^{n+\infty}
\]

\[
= \frac{CF}{r} + \frac{d}{r^2} - \left[ \frac{CF_n + d}{r} + \frac{d}{r^2} \right] \left[ \frac{1}{(1+r)^n} \right]
\]

where \( CF_n \) is the cash flow at point \( n \).

\[
= \frac{CF}{r} + \frac{d}{r^2} - \left[ \frac{CF + (n-1)d}{r} + \frac{d}{r^2} \right] \left[ \frac{1}{(1+r)^n} \right]
\]

\[
= \left( \frac{CF}{r} \right) \left( \frac{1}{r} \right) \left( \frac{1}{1+r} \right)^n - \frac{(n-1)d}{r} \left( \frac{1}{1+r} \right)^n \ldots \ldots \ldots (10.12)
\]
3. Two stage Arithmetic growth
Assuming that the period growth rate is $d_1$ for period 1 to n-1 (first stage) and after that it grows at a growth rate $d_2$ for period n to $\infty$ (second stage). Its present value at time 0 can be derived as

$$PV_0 = \frac{CF}{r} + \frac{d_1}{r^2} - \frac{\left[ \frac{CF_{n-1} + d_1 + d_1}{r} \right]}{(1+r)^{n-1}} + \frac{\left[ \frac{CF_{n-1} + d_2 + d_2}{r} \right]}{(1+r)^{n-1}}$$

$$= \frac{CF}{r} + \frac{d_1}{r^2} + \frac{\left( d_2 - d_1 \right) + \left( d_2 - d_1 \right)}{r^2} \left( 1 + \frac{1}{r} \right)^{n-1}$$

$$\text{Illustration:}$$
Mr. Stevenson will deposit $100 in an account at the end of first year. He expects to deposit extra $10 each year in the account for next 9 years. Beginning 11th year he will increase the extra deposit by $20 per year for forever. Assuming rate of interest of 5% compounded annually, what will be the present value of these deposits?

We draw the time line and use the above derived formula (10.13) to find out the present value.

<table>
<thead>
<tr>
<th>$100$</th>
<th>$110$</th>
<th>$190$</th>
<th>$210$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>11</td>
</tr>
</tbody>
</table>

Substituting the values in expression (10.13), we have,

$$= \frac{100}{.05} + \frac{10}{(0.05)^2} + \frac{\left( 20 - 10 \right)}{.05} + \frac{\left( 20 - 10 \right)}{(0.05)^2} \left( 1 + .05 \right)^{10}$$

$$= 6000 + 1924.06 = \$ 7924.06$$

4. Multi-stage Arithmetic growth
If we increase the number of stages from two to many, the formula for finding present value will be

$$PV = \left( \frac{CF}{r} + \frac{d_1}{r^2} \right) + \sum_{t=1}^{n} \frac{\left( d_{t+1} - d_t \right) + \left( d_{t+1} - d_t \right)}{r^2} \left( 1 + r \right)^{n(t)}$$
Section V
Application of Arithmetic and Geometric Growth

1. Life Cycle Data:
A good example of piece-wise arithmetic growth and arithmetic growth in general is the literature on life cycles. The product life cycle is characterized by 4 phases of growth – introduction, growth, maturity and decline in terms of product sales. The product life cycle diagram is identified below

![Product Life Cycle Diagram](image)

This offers a good example of two stage arithmetic growth by broken lines. Alternatively, project expansion literature identifies examples of arithmetic growth as expansion or project development is initiated at the marginally most productive site or opportunity (highest growth rate) and then continued overtime to marginally less productive sites or opportunities, i.e., those contributing less growth as illustrated in the graph below.

![Growth Rate in Cash Flows](image)

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Cash flow modeled for the diminishing growth rates are proxied very well with arithmetic growth and less accurately with geometric growth models.

2. Simple Macaulay’s Duration

The Macaulay’s Duration is given by

\[
D = \frac{\sum_{i=1}^{n} tc \left(\frac{1}{1+r}\right)^i + \frac{nF}{1+r} - K + \frac{nF}{(1+r)^n}}{p} = \frac{K + \frac{nF}{(1+r)^n}}{p} \quad \text{(10.15)}
\]

Where
- \( t = \) weights 1, 2, \ldots, \( n \)
- \( c = \) coupon amount
- \( r = \) Yield to maturity
- \( F = \) maturity value
- \( p = \) present price
- \( K = \sum_{i=1}^{n} \frac{tc}{(1+r)^i} \)

K can be written as;

\[
K = \frac{c}{1+r} + \frac{2c}{(1+r)^2} + \frac{3c}{(1+r)^3} + \ldots + \frac{nc}{(1+r)^n} \quad \text{(10.16)}
\]

Multiplying both sides by \( \frac{1}{1+r} \) we have

\[
\frac{K}{1+r} = \frac{c}{(1+r)^2} + \frac{2c}{(1+r)^3} + \ldots + \frac{(n-1)c}{(1+r)^n} + \frac{nc}{(1+r)^{n+1}} \quad \text{(10.17)}
\]

Subtracting (10.17) from (10.16) we have

\[
K - \frac{K}{1+r} = \frac{c}{1+r} + \frac{c}{(1+r)^2} + \ldots + \frac{c}{(1+r)^n} - \frac{nc}{(1+r)^{n+1}}
\]

\[
K^{1+r} = c \left[ \frac{1 - \left( \frac{1}{1+r} \right)^n}{1 - \frac{1}{1+r}} \right] - \frac{nc}{(1+r)^{n+1}}
\]

\[
K^{r} = c \left[ \frac{1 - (1+r)^n}{r} \right] - \frac{nc}{(1+r)^{n+1}}
\]
or $K = \frac{1+r}{r^2}c[1-(1+r)^{-n}] - \frac{nc}{(1+r)^{n+1}} \frac{1+r}{r} \ldots \ldots (10.18)$

Substituting (18) in (15) we have

$$D = \frac{\frac{1+r}{r^2}c[1-(1+r)^{-n}] + \frac{n}{(1+r)^n} \left[ F - \frac{c}{r} \right]}{p}$$

The first component of the numerator is exactly the present value of arithmetic growth which is easily expressed with the arithmetic growth closed form solution. When applied to durations of annuities, par bonds, etc., simple closed form solutions to duration result from the use of the arithmetic present value formulation.

### 3. Forensic Economics and Age earning profiles

Age earning profiles by level of education are characterized by this concave illustration below

The forensic economics literature is full of models in which geometric growth is utilized in solving economic loss associated with injury or wrongful death to individuals of different ages. They either omit the age earning profiles identified above or mis-specify the profiles with a constant geometric growth model. Clearly, the arithmetic model offers a more accurate model for valuation of loss over most ages of injured parties.