Problems

1. In how many years $100 will become $265 if \( k = 11\% \)?
\[
n = \frac{\ln \left( \frac{265}{100} \right)}{\ln(1 + 0.11)} = 9.33844 = 9.34 \text{ years}
\]

2. In how many years will an amount double if \( k = 7.6\% \)?
\[
n = \frac{\ln 2}{\ln 1.076} = 9.46 \text{ years}
\]

3. In how many years will an amount triple if \( k = 9\% \)?
\[
n = \frac{\ln 3}{\ln 1.09} = 12.75 \text{ years}
\]

4. At what rate of interest $100 will become $265 in 20 years?
\[
k = \left( \frac{265}{100} \right)^{\frac{1}{20}} - 1 = 0.4993 = 4.99\%
\]

5. At what rate of interest an amount will double in 15 years?
\[
k = (2)^{\frac{1}{15}} - 1 = 0.4729 = 4.73\%
\]

6. At what \( k \) an amount will triple in 6 years?
\[
k = (3)^{\frac{1}{6}} - 1 = 0.20094 = 20\%
\]

7. We are going to receive $200 at the beginning of the 30\text{th} year. If \( k = 0.3\% \), what is the present value?
\[
PV = \frac{300}{(1 + .003)^{29}} = $275.04
\]
8. You will receive $300 in the 25th year. What is the present value at the beginning of year 12 if \( k = 15\% \)?

\[
PV = \frac{300}{(1 + .15)^{14}} = $42.39860
\]

9. We receive $300 in the 15th year. We will put that money in an account earning \( k = 5\% \). What will the account be worth at the 40th year?

\[
PV = \frac{300}{(1.05)^{15}} = $144.30513 = $144.31
\]

\[
FV = 300(1.05)^{25} = $1,015.90648 = $1,015.91
\]

\[
FV_{40} = 144.31 + 1,015.91 = $1,160.22
\]

10. What is the future value of $15 in year 4 if \( k = \)?

\[
FV = 15(1.10)(1.15)(1.12)(1.16) = 24.65232 = $24.65
\]

11. Obtain the future value of $10 if \( k = -10\% \) for the first 5 years, next five years \( k = 11\% \) and next 3 years 13% in 13 years.

\[
FV = 10(1.10)^5(1.11)^5(1.13)^3 = $39.16
\]

12. In how many years an account will double at a given rate of interest? \( k = 5\% \)

(Use Rule 72)

\[
n = \frac{72}{(100)(0.05)} = 14.4
\]

13. At what rate of interest an account will double in 20 years?

\[
k = \frac{72}{(100)(20)} = 0.036
\]

\[
k = 3.6\%
\]

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Problems - 2
14. Assume that $k$ during the first 3 years is 10%, next 3 years is 20%, and next 6 years is 12%. If you deposit $100 today, how much will you have in 11 years?

$$ FV = 100(1.10)^3(1.20)^3(1.12)^5 = 405.33 $$

15. We receive $100 at the beginning of 50th year. If $k = 6\%$ for first 20 years, 12% next 10 years and remaining period is 16%. What is the PV of the amount?

$$ PV = \frac{100}{(1.06)^{20}(1.12)^{10}(1.16)^{10}} = 0.59841 = 0.60 $$

16. We receive $100 at the end of year 10. If $k = 0\%$ what is the PV today?

$$ PV = \frac{100}{(1.00)^{10}} = \frac{100}{1} = 100 $$

17. In how many years $24 will become $2400 if $k = 300\%$?

$$ n = \frac{\ln \left( \frac{2400}{24} \right)}{\ln(1 + 3)} = 3.2\,\text{years} $$

18. The current price of the island of Manhattan is $5\text{trillion}$. 275 years ago the island was sold by native Indians for $24. If $k = 15\%$ per year, was the island sold cheaply?

$$ FV = 24(1.15)^{275} = 1.18064\times10^{18} $$

Yes, the island was sold too cheaply !!

19. Obtain the PV of deposits of $10, $20, $ 30 in years 1, 2, 3 if rate of interest is 5 \%

$$ PV = \frac{10}{(1.05)} + \frac{20}{(1.05)^2} + \frac{30}{(1.05)^3} = 9.52 + 18.14 + 25.92 = 53.58 $$

20. Obtain the PV of deposits of $10, $20, $ 30 in beginning of years 1, 2, 3 if rate of interest is 5 \%

$$ PV = 10 + \frac{20}{(1.05)} + \frac{30}{(1.05)^2} = 10 + 19.05 + 27.21 = 56.26 $$
21. Obtain the PV of deposits of $10, $20, $30
   a. deposits at end of the year
   b. in beginning of years 1, 2, 3 if simple rate of interest is 5%

   a. \[ PV = \frac{10}{1 + (.05)(2)} + \frac{20}{1 + (.05)(3)} + \frac{30}{1 + (.05)(3)} = 9.52 + 18.18 + 26.09 = \$53.79 \]
   b. \[ PV = 10 + \frac{20}{1 + (.05)(2)} + \frac{30}{1 + (.05)(2)} = 10 + 19.05 + 27.27 = \$56.32 \]

22. You deposit $10 in an account every other year for 6 years. If \( k = 10\% \), what is the
   PV of these deposits assuming
   a. simple rate of interest
   b. compound rate of interest

   a. \[ PV = \frac{10}{1 + (.10)(2)} + \frac{10}{1 + (.10)(4)} + \frac{10}{1 + (.10)(6)} = 8.33 + 7.14 + 6.25 = \$21.72 \]
   b. \[ PV = \frac{10}{(1.10)^2} + \frac{10}{(1.10)^4} + \frac{10}{(1.10)^6} = 8.26 + 6.83 + 5.65 = \$20.74 \]

23. You deposit $10 in an account every other year for 6 years. If \( k = 10\% \), what is the
   FV of these deposits assuming
   a. simple rate of interest
   b. compound rate of interest

   a. \[ FV = 10\left[1 + (4)(.10)\right] + 10\left[1 + (2)(.10)\right] + 10 = 14 + 12 + 10 = \$36 \]
   b. \[ FV = 10(1.10)^4 + 10(1.10)^2 + 10 = 14.64 + 12.10 + 10 = \$36.74 \]

24. Consider the following series
   \[ S = 1 + 4^1 + 4^2 + 4^3 + 4^4 + 4^5 + \ldots + \infty \]
   \[ S = \infty \] because common ratio (4) is > 1.
Practice Problems {Solutions Appear Below}

1. You deposit a lump sum of $5,000 in an account today. Assuming the following interest rate structure: K=6% for the first 10 years, 8% for the second 10 years and 10% for the last 10 years. What is the future value of the deposit after 30 years if the account offers
   (a) simple interest rates?
   (b) annually compounded interest rates?
   (c) monthly compounded interest rates?
   (d) continuously compounded interest rates?

2. You expect to receive $1,000,000 ten years from today. Suppose the interest rates will be 8% for the first 5 years and 10% for the next five years, what is the present value if we have
   a) simple interest rates?
   b) annually compounded interest rates?
   c) monthly compounded interest rates?
   d) continuously compounded interest rates?

3. You have the choice between two accounts A pays 8% compounded continuously and account B pays 8.1% compounded semiannually. Which one offers a better deal and why?

4. What is the actual yield (or APR) of (a) 15% compounded quarterly? (b) 18% compounded monthly (c) 12% compounded semi-annually (d) 10% compounded continuously?

5. What is the equivalent nominal quarterly compounded rate of
   (a) 8% compounded annually?
   (b) 10% compounded monthly?
   (c) 12% compounded semiannually?
   (d) 10% compounded continuously?

6. What is the equivalent nominal continuously compounded rate of interest of
   (a) 8% compounded annually?
   (b) 10% compounded monthly?
   (c) 12% compounded semiannually?
   (d) 10% compounded quarterly?

7. Suppose you deposit $1,000 at the end of each year for 20 years. Find the FV at the end of the 30th year with
   (a) 10% simple interest rate?
   (b) 10% annually compounded interest rate?
8. Suppose you deposit $1,000 at the end of each year for 10 years. Find the present value of the annuity with
   (a) 10% simple interest rate?
   (b) 10% annually compounded interest rate?
   (c) 10% continuously compounded interest rate?

9. You deposit $5,000 in an account that pays 6% compounded annually for 6 years and 8% thereafter. What will be your balance after 15 years?

10. What is the present value (cash value) of an endowment of $100,000 that you expect to receive at your retirement at age 65? You are 40 years old today. The interest rate is expected to be 8%. What will be the cash value if interest rates are expected to be 5%?

11. You expect to pay $50,000 for your child’s education when she is 18. How much must you deposit when the child is 2 years old to insure availability? K=18%.

12. You deposit $7,000 in an account today. What is the future value after 25 years with
   (a) 8.5% simple interest
   (b) 8.5% interest compounded annually
   (c) 8.5% interest compounded quarterly
   (d) 8.5% interest compounded continuously. What can you imply from this exercise?

13. What is the future value of $5,000 per year after 15 years with
   (a) 9.5% simple interest
   (b) 9.5 interest compounded annually
   (c) 9.5% interest compounded continuously?

14. What is the present value of $2,000 per year paid for 8 years with
   (a) 9% simple interest
   (b) 9% interest compounded annually
   (c) 9% interest compounded continuously. What can you imply from this exercise?

15. What is the effective annual rate (EAR) of
   a) 15%, 18% compounded quarterly
   b) 12%, 18% compounded monthly
c) 9% and 11% compounded continuously?

For questions 16 - 17 find the present value of the cash flows. K=8%.

16. $6,000/year for 12 years and then perpetual growth of 2%/year.

17. $9,000/year for 5 years and then negative growth of 4.5%/year for another 6 years.

18. Miami Autos offers a loan at 10% on a new $10,000 listed car. You pay $1,000 down and then $300 a month for the next 30 months. Kendall Autos does not offer a loan but will give you $1,000 off the list price. Which company is offering the better deal?

19. You own an oil pipeline that generates $2 million cash flow over the next year. The pipeline’s operating costs are negligible and it’s is expected to last for a very long time. Unfortunately, the volume of oil shipped is declining and cash flows are expected to decline by 4% per year. K=10%.
(a) What is the PV of the cash flow if it is assumed to last forever?
(b) What is the PV of the cash flow if the pipeline is scrapped after 20 years?

20. Your firm expects to earn $200,000 this year and projects a growth rate in earnings of 5%/year thereafter K=10%. What is the PV of the earnings if:
(a) It expects growth forever.
(b) It expects growth for another 10 year and then it shuts down.
(c) It expects growth for another 10 years and then level earnings forever.

21. You have made the following proposition: “Pay us $200/year for 10 years and we will pay you $200/year thereafter in perpetuity”. Is this a fair deal, what is the interest rate?

22. ABC corp. borrows $50 million for 20 years at 12%. It is required to set up an escrow account to pay off the principal at maturity. What will be the initial annual contribution to the fund with the following stipulations
(a) level payments each year.
(b) payments grow at 6% per year (find payments 1 and 2).

23. You need to set up a college fund at birth for your child that will pay $60,000 at age 18. K=12%. (a) How much must you deposit at the end of each year with annual compounding (b) How much must you deposit at the end of each month with monthly compounding (c) What will be the payments if they are at the beginning of the year (18 payments)?

24. You contributed $2,400 per year for 40 years and then retired. You expect to live another 20 years K=10%. What will be your annual pension check after retirement? Suppose you made
the premium payments and collected the pension checks at the beginning of each year. What will be your annual pension check?

25. You contributed $3,500 per year for 40 years and the retired. You expect to live another 25 years. What will be your annual pension check with (a) 9% interest compounded annually (b) 9% compounded continuously.

26. You must choose between two projects. Project A expects to earn nothing for 2 years, $5 million in the 3\textsuperscript{rd} year and then grow at 4%/year for another 8 years and then level earnings forever. Project B expects to earn nothing for three years, $6 million in the 4\textsuperscript{th} year, continue to earn $6m/year for another five years, and then have perpetual growth of 2%/year. K=10%.

27. ABC corp. expects to earn $600,000 this quarter and expects growth of 6% per year. It has a dividend payout ratio of 1/3 and plans to use the remainder to payoff loans in quarterly installments. K=12%. It needs to finance a project a 10 year loan. How much can it afford to borrow now?

28. You set up a college fund for your child that will pay $100,000 at age 18. You deposit $5,000 up front and another $5,000 after 10 years. K=10%. How much in addition do you need to deposit each year to accumulate the required amount with (a) annual (b) monthly (c) continuous compounding.

29. You deposit $1,000 at 9%. How long will it take to double, triple with (a) annual (b) quarterly (c) continuous compounding.

30. You deposit $100/month at 10%. How long will it take for the balance to reach $5,000 with monthly compounding?

31. You inherited a lumpsum of $500,000 and put it into an account at 10%. How long will the amount last if you with draw (a) $25,000/year (b) 75,000/year.

32. You deposit $7,000 lumpsum in an account. You wish to close the account and with draw the money when the balance reaches $15,000. How long will it take if the interest rate is (a) 12% compounded annually (b) 12% compounded monthly.

33. You invested $6,000. In six years you get back $15,000. What is the interest rate with (a) annual (b) monthly (c) continuous compounding.

34. Investment A requires $10,000 and promises $18,000 after 8 years. Investment B promises 7.5% continuously compounded return. Which one should you choose?

35. You take out a $100,000 mortgage at 7.2% for 30 years. Your monthly payment is $678.80. However, you decide to pay an extra $100 per month. How long will take to pay off the mortgage?

36. You take a $200,000 mortgage at fixed rate for 30 years. What are the monthly payments if (a) interest rates are 7.8%, (b) interest rates are 9%.
37. You take a $100,000 mortgage at fixed rate of 8.4%. What are the monthly payments if the mortgage is for (a) 30 years (b) 15 years?

Solution To The Problems 1 To 37

1. (a) \( FV = 5,000 \left[ 1 + \left( \frac{0.06}{12} \right) (10) + \left( \frac{0.08}{12} \right) (10) + \left( \frac{0.10}{12} \right) (10) \right] = 17,000 \)

(b) \( FV = 5,000 \left( 1.06 \right)^{10} \left( 1.08 \right)^{10} \left( 1.1 \right)^{10} = 50,141.10 \)

(c) \( FV = 5,000 \left[ 1 + \left( \frac{0.06}{12} \right) \right]^{120} \left[ 1 + \left( \frac{0.08}{12} \right) \right]^{120} \left[ 1 + \left( \frac{0.10}{12} \right) \right]^{120} = 54,660.66 \)

(d) \( FV = 5,000 e^{\left( 0.06 \right) (10)} e^{\left( 0.08 \right) (10)} e^{\left( 0.10 \right) (10)} = 5,000 e^{2.4} = 55,115.88 \)

2. (a) \( PV = \frac{1,000,000}{\left[ 1 + \left( \frac{0.08}{12} \right) \right]^{120} \left[ 1 + \left( \frac{0.10}{12} \right) \right]^{120}} = 526,315.79 \)

(b) \( PV = \frac{1,000,000}{\left( 1.08 \right)^{5} \left( 1.10 \right)^{5}} = 422,588.62 \)

(c) \( PV = \frac{1,000,000}{\left[ 1 + \left( \frac{0.08}{2} \right) \right]^{20} \left[ 1 + \left( \frac{0.10}{2} \right) \right]^{20}} = 407,954.05 \)

(d) \( PV = \frac{1,000,000}{e^{\left( 0.08 \right) (5)} e^{\left( 0.10 \right) (5)}} = \frac{1,000,000}{e^{0.9}} = 406,569.66 \)

3. (a) Account A has APR = \( e^{0.08} - 1 = 8.33\% \)

Account B has APR = \( e^{\left( \frac{0.08}{2} \right)} \left( \frac{12}{2} \right) \left( \frac{10}{2} \right) \left( \frac{5}{2} \right) = 8.264\% \)

Therefore the answer is account A.

4. (a) \( \left( 1 + \left( \frac{0.15}{4} \right) \right)^{4} - 1 = 15.865\% \)

(b) \( \left( 1 + \left( \frac{0.18}{12} \right) \right)^{12} - 1 = 19.562\% \)

(c) \( \left( 1 + \left( \frac{0.12}{2} \right) \right)^{2} - 1 = 12.36\% \)

(d) \( e^{0.10} - 1 = 10.517\% \)

Hint: use \( K_a = \left( 1 + \left( \frac{K_m}{m} \right) \right)^m - 1 \)

5. Use both \( K_m = m\left[ (1+KJ)^{1/m-1} \right] \) and \( K_a = \left[ 1 + \left( K_m/m \right) \right]^{m-1} \)

(a) \( K_a = 8\%, K_m = 4\left[ 1 + \left( 0.08 \right) \right]^{1/4} - 1 = 7.77\% \)

(b) \( K_{12} = 10\%, K_a = \left[ 1 + \left( 0.10/12 \right) \right]^{12} - 1 = 10.4713\% \)

(c) \( K_{12} = 12\%, K_a = \left[ 1 + \left( 0.12/2 \right) \right]^{2} - 1 = 12.36\% \)

(d) \( K_{10} = 10\%, K_a = e^{0.10} - 1 = 10.517\% \)

6. Use both \( K_a = e^{K_c \cdot 1} \) and \( K_c = \ln(1 + K_a) \)

(a) \( K_a = 8\%, K_c = \ln(1 + 0.08) = 7.6961\% \)

(b) \( K_{12} = 10\%, K_c = \ln(1 + 0.10/12) = 9.9586\% \)

(c) \( K_{12} = 12\%, K_c = \ln(1.1236) = 11.6538\% \)

(d) \( K_{10} = 10\%, K_c = \ln(1 + 0.10/4) = 9.877\% \)

7. A=1,000 n=20

(a) \( K_a = 10\% \) simple interest rate. \( PVA = \frac{nA}{1 + (n-1)k/2} \)

\( FV = 1,000(20) \left[ 1 + \left( \frac{20}{2} \cdot (0.10) \right) \right] = 59,000 \)

(b) \( K_a = 10\% \)

\( FV = 1,000 \left[ (1.1)^{20} - 1 \right]/0.10(1.1)^{10} = 148,556.60 \)

(c) \( K_a = 10\% \), \( FVA = A\left[ e^{K_c \cdot 1} / e^{K_c} \right] \)

\( FV = 1,000 \left[ \frac{e^{0.10}}{1} - 1 \right] \left( \frac{1}{e^{0.10}} \right) e^{0.10} = 165,133.63 \)

8. A=1,000 n=10

(a) \( K_a = 10\% \) simple interest rate. \( PVA^e = \frac{A}{1+k} + \frac{A}{1+2K} + \ldots + \frac{A}{1+10K} \)
(b)  
\[ \text{PV} = \frac{1000}{1.1} + \frac{1000}{1.2} + \frac{1000}{1.3} + \ldots + \frac{1000}{1.9} + \frac{1000}{2} = 6,687.73 \]

(c)  
\[ \text{PV} = 1000 \left( \frac{1 - (1.1)^{-10}}{0.1} \right) = 6,144.5 \]

(c)  
\[ \text{PV} = 1000 \left( \frac{1 - (1.1)^{-10}}{0.1} \right) \left( \frac{1}{e^{0.10} - 1} \right) = 6,010.41 \]

9.  
\[ F_{15} = 5,000(1.06)^{15} = 14,786.13 \]

10.  
\[ K = 8\% \text{ PV} = 100,000/(1.08)^{25} = 14,601.79 \]

\[ K = 5\% \text{ PV} = 100,000/(1.05)^{25} = 29,530.28 \]

11.  
Amount to be deposit = PV of future requirement = 50,000/(1.08)^{16} = 14,594.52

12.  
(a)  
\[ \text{FV} = A(1+nK) = 7,000 \left( 1 + 25 \times 0.085 \right) = 21,875 \]

(b)  
\[ \text{FV} = A(1+K) = 7,000 \left( 1.085 \right)^{25} = 53,807.33 \]

(c)  
\[ \text{FV} = A \left( 1 + \left( \frac{K}{m} \right) \right)^{mn} = 7,000 \left( 1.02125 \right)^{100} = 57,320.84 \]

(d)  
\[ \text{FV} = A e^{Kn} = 7,000 e^{2.125} = 58,610.2 \]

FV is higher with compound interest and as \( m \) increases, FV decreases. It is higher with continuous compounding.

13.  
(a)  
\[ \text{FVA with simple interest} = 5,000 \left( 1 + 14 \times 0.095/2 \right) = 124,875 \]

(b)  
\[ \text{FVA with annual compounding} = 5000 \left[ \frac{(1.095)^{15} - 1}{0.095} \right] = 57,320.84 \]

(c)  
\[ \text{FVA with continuous compounding} = 5000 \left[ \frac{e^{1.425} - 1}{e^{0.095} - 1} \right] = 58,610.2 \]

14.  
(a)  
\[ \text{PVA with simple interest} = 2000/1.09 + 2000/1.18 + 2000/1.27 + \ldots + 2000/1.72 = 11,643 \]

(b)  
\[ \text{PVA with annual compounding} = 2000 \left[ \frac{1 - (1/1.09)^{8}}{0.09} \right] = 11,069 \]

(c)  
\[ \text{PVA with continuous compounding} = 2000 \left[ \frac{1 - (1/e^{0.72})}{e^{0.09} - 1} \right] = 10,900 \]

As \( m \) increases, PVA decreases. Also PVA is greater with simple interest. 

15.  
(i)  
\[ \text{K_a} = \text{EAR} = \left( 1 + \left( \frac{K}{m} \right) \right)^m - 1 = (1.0375)^4 - 1 = 0.15865 = 15.865\% \] and equally 18% compounded quarterly is equivalent to EAR of 19.25%.

(ii)  
For 12% compounded monthly \( \text{K_a} = \text{EAR} = \left( 1 + \left( \frac{0.01}{12} \right) \right)^{12} - 1 = 0.1268 = 12.68\% . \)

For 18% compounded monthly \( \text{K_a} = \text{EAR} = \left( 1.015 \right)^{12} - 1 = 19.56\% . \)

(iii)  
For 9% compounded continuously \( \text{K_a} = \text{EAR} = e^{0.09} - 1 = 9.42\% . \)

For 18% compounded continuously \( \text{K_a} = \text{EAR} = e^{0.18} - 1 = 19.72\% . \)

16.  
\[ \text{PV} = 6,000 \left[ \frac{1 - (1/1.08)^{12}}{0.08} \right] + \left[ \frac{(6,000)(1.02)}{(0.08-0.02)} \right] / (1.08)^{12} = 45,216.5 + 40,505.6 \]
\[ = 85,722.071 \]

\[ \text{PV} = \text{PV of C,1-12} + \text{PV of C,13-\infty} \]

17.  
\[ 1+g = 1 + 0.095 \]

\[ \text{PV} = 9,000 \left[ \frac{1 - (1/1.08)^{5}}{0.08} \right] + 9,000(0.955) \left[ \frac{(1-(0.955/1.08)^{6})}{(0.08-0.045)} \right] / (1.08)^5 \]
\[ = 60,359.81 \]

18.  
\[ \text{K/m} = 0.008333. \text{PV of payments to Miami Auto} = 1000 + 300(PVAF) \]

\[ \text{PV of Miami Auto Payments} = 1,000 + 300[(1-(1/1.008333)^{30})/0.008333] = 8,934.152 \]

\[ \text{PV of Kendall Auto Payments} = 9,000 \text{ cash. So the better deal will be with Miami Auto.} \]

19.  
(a)  
\[ \text{PV} = C/(k-g) = 2/0.10 - (-0.04) = 2/0.14 = 14.29 \text{ million} \]

(b)  
\[ \text{PV} = 2(PVAFC) = 2 \left[ \frac{(1-(0.96/1.1)^{20})}{0.14} \right] = 13.34713 \text{ million} \]

20.  
(a)  
\[ \text{PV} = C/(k-g) = 200,000/(0.10-0.05) = 4,000,000 \]

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(b) \[ PV = 200,000 \frac{1-(1.05/1.1)^{11}}{0.10-0.05} = 1,602,145.96 \]

(c) Earnings for the 11\textsuperscript{th} year = 200,000 \times 1.05^{10} = 325,778.93 \text{ This are also the earnings for years 12 to } \infty;

\[ PV = 1,602,145.96 + \frac{325,778.93/0.10}{(1.1)^{11}} = 1,602,145.96 + 1,141,835.27 \]
\[ = 2,743,981.23 \]

21. Find K. PV of 200/year for 10 years = 200 \frac{1-(1/(1+K))^{10}}{K} \text{ then } PMT = 50,000,000/72.05 \text{ PMT } = 693,962.53

(b) FVAC = 50,000,000 = P \frac{1}{\frac{((1+K)^n-(1+g)^n)}{(k-g)}} = P \frac{1}{107.3193} = 50,000,000/107.3193 = 465,899.4234

(c) FVAC due at age 18 = 60,000 = P \frac{1}{\frac{((1.12)^{18}-(1+g)^{18})}{(k-g)}} = P \frac{1}{93.1193} = 536,945.6170

22. (a) FVA = 50,000,000 = P \times FVAF = P \left[ \frac{((1.12)^{20} -1)}{0.12} \right] \text{ then } PMT = 50,000,000/72.05 \text{ PMT } = 693,962.53

(b) FVA due = P \frac{1}{\frac{((1+K)^n-(1+g)^n)}{(k-g)}} = P \frac{1}{107.3193} = P \frac{1}{93.1193} = 50,000,000/93.1193 = 536,945.6170

23. (a) FVA = 60,000 = P \times FVAF = P \left[ \frac{((1.12)^{18} -1)}{0.12} \right] \text{ then } PMT = 60,000/55.7497 \text{ PMT } = 1076.24/year

(b) With monthly compounding k/m=0.01, mn=216 months

FVA = 60,000 = P \left[ \frac{((1.01)^{216} -1)}{0.01} \right] = P \left( 757.8606 \right) \text{ P = 60,000/757.8606 = 79.1702/year}

(c) FVA due at age 18 = 60,000 = P \left[ \frac{((1.01)^{216} -1)}{0.01} \right] \text{ (1.12) } = 62.4397

FVA monthly payments due = 60,000 = P \left[ \frac{((1.01)^{216} -1)}{0.01} \right] \text{ (1.01) } = 765.4392

P = 60,000/765.4392 = 78.386/month

24. (a) FVA of premium payments = PV of annuity of pension checks at t=40 (at retirement).

FVA\textsubscript{40} = 2,400 \left[ \frac{1-(1.1)^{40}}{0.10} \right] = 1,062,222.1336 = PVA = P \left[ \frac{1-(1.1)^{40}}{0.10} \right] \text{ PVA } = \frac{1,062,222.1336}{124,768.2132} = 8.51356

(b) FVA\textsubscript{due} = PVA\textsubscript{due}

2,400 \left[ \frac{1-(1.1)^{40}}{0.10} \right] \text{ (1.1) } = P \left[ \frac{1-(1.1)^{40}}{0.10} \right] \text{ (1.1) } = 124,768.2132 still the same

25. (a) FVA premium =3,500 \left[ \frac{1-(1.09)^{40}}{0.09} \right] = PVA of checks = P \left[ \frac{1-(1.09)^{40}}{0.09} \right] \text{ Annual pension check } = P = 1,182,588.558/9.82258 = 120,394.906/year

(b) FVA with continuous compounding = 3,500 \left[ \frac{\exp(0.09) - 1}{\exp(0.09) - 1} \right] = 1,323,013.20

PVA = P \left[ \frac{1-(1/\exp(0.09))^{40}}{\exp(0.09) - 1} \right] = 9.499416
26. PV of project A = \(5\left(\frac{1-(1.04/1.1)^9}{0.10-0.04}\right)/(1.1)^9\) + \((5 \times (1.04)^9)/0.10\)/(1.1)^{11}\) 
\[= 51.2824 \text{ million}\] 
PV of project B = \(6\left(\frac{1-(1/1.1)^{10}}{0.10-0.02}\right)/(1.1)^{9}\) + \(6 \times (1.02)/0.10\)/(1.1)^{11}\) 
\[= 52.0764 \text{ million}\] 
The best choice is project B because it has a higher PV.

27. Amount available for repayment in the 1st quarter = \(\frac{2}{3} \times 600,000 = 400,000\) 
\[mn=40 \ g=0.06/4=0.015/\text{quarter} \ k/m=0.12/4=\] 
PV = \(400,000 \left[\frac{1-(1.015/1.03)^{40}}{0.03-0.015}\right] = 11,837,339.9341\).

28. (a) FV of 5,000 deposit up front = \(5,000 \times (1.1)^{18}\) = \(27,799.59\) at age 18. 
FV of 5,000 deposit at age 10 = \(5,000 \times (1.1)^8\) = \(10,717.94\) at age 18. 
Amount required = \(100,000 - 27,799.59 - 10,717.94 = 61,482.47\) 
FVA = \(61,482.47 = \text{PMT} \times \text{FVAF} = \frac{P \left((1.1)^{18}-1\right)}{0.10}\) 
P = \(61,482.47 / 45.5992 = 1348.32 / \text{year}\).

(b) FVA of 5,000 up front = \(5,000 \times (1.008333)^{216}\) = \(30,023.467\) at age 18 
FVA of 5,000 deposit at age 10 = \(5,000 \times (1.008333)^{96}\) = \(11,090.878\) 
Amount required = \(58,885.655\) 
P = \(\frac{\text{FVA}}{\text{FVAF}} = \frac{58,885.655}{((1.008333)^{216}-1)/0.00833}\) = \(98.0551\)
(c) With continuous compounding the remaining requirement = \(100,000 - 5,000e^{\left(0.10\right)\left(18\right)} - 5,000e^{\left(0.10\right)\left(18\right)}\) 
\[= 58,624.06\] 
P = \(58,624.06/\left(e^{\left(0.10\right)\left(18\right)}-1\right)/\left(e^{\left(0.10\right)-1}\right) = 1220.98 / \text{year}\).

29. (a) \(n = (\ln (\frac{F}{A})) / \ln(1+k) = \ln(2)/\ln (1.09) = 8.043\) years to double with annual compounding 
\(n = \ln(3)/\ln(1.09) = 12.7482\) years to triple with annual compounding 
(b) \(n = \ln (F/A)/ m\ln(1+k) = \ln(2)/4\ln(1.0225) = 7.7879\) years to double with quarterly compounding. 
\(n = \ln(3)/4\ln(1.0225) = 12.3436\) years to triple with quarterly compounding. 
(c) \(n = \frac{\ln(F/A)/k}{\ln(3)/0.09} = 12.2068\) years to double with quarterly compounding.

30. (a) \(n = \left(\ln[1+(k/m)\times(F)/P]\right)/(m\ln[1+(k/m)]) = \left(\ln [1+0.00833(5000/100)]\right)/ \left(12\ln(1.00833)\right)\) 
\[= (\ln 1.4167)/12\ln(1.00833) = 3.4978\) years.

31. \(n = (-\ln(1-\frac{\text{AP}}{\text{PK}}))/(\ln(1+k))\)
(a) It will last forever because \(\text{AK} > \text{P}\) where AK is interest inflow and P is the withdraw. 
\(50,000>25,000\).
(b) With \(P=75,000\) the inherited lumpsum will last 11.5267 years.

32. \(A=7,000 \ F=15,000\)
(a) \(n = (\ln (15,000/7,000))/(\ln(1.12) = 6.7250\) years with annual compounding. 
(b) \(n = (\ln (15,000/7,000))/12(\ln 1.01) = 6.3829\) years with monthly compounding.

33. \(A=6,000 \ F=15,000 \ n=6\) years
(a) \(k = (15,000/6,000)^{1/6} - 1 = 0.16499 = 16.50\%\)
(b) \(k = m[(15,000/6,000)^{1/72}-1] = 12(0.01280) = 15.3691\%\)
(c) \( k = (1/n)(\ln(F/P)) = (1/6)(\ln(15,000/6,000)) = 0.1819 = 18.19\% \)

34. Investment A is paying: \( K = (\ln(18,000/10,000))/8 = 0.07347 = 7.3473\% \) compounding continuously.

Investment B promises 7.5% continuously compounding return.

Then the best deal is Investment B.

35. \( A = 100,000 \quad k/m = 0.072/12 = 0.006 \quad P = 678.80 + 100 = 778.80 \)

\[
n = \frac{-\ln(1 - ((A(k/m))/P))}{m \ln(1+(k/m))} = \frac{-\ln(1 - (100,000*0.006)/(778.80))/(12 \ln(1.006))}{12 \ln(1.006)}
\]

\( = 20.4985 \) years.

36. (a) \( A = 200,000 \quad mn = 30*12 = 360 \) months \( k/m = 0.078/12 = 0.0065 \) MP = principal plus interest

\[
MP = \frac{A}{PVAF} = \frac{200,000}{[(1-(1/1.0065)^{360})/0.0065]} = 200,000/138.91387 = 1,439.74/month
\]

(b) \( MP = 200,000/[(1-(1/1.0075)^{360})/0.0075] = 200,000/124.2818 = 1,609.2460/month
\]

where \( k/m = 0.09/12 = 0.0075 \)

37. (a) \( A = 100,000 \quad k/m = 0.084/12 = 0.007 \quad mn = 30*12 = 360 \) months

\[
MP = \frac{A}{PVAF} = \frac{100,000}{[(1-(1/1.007)^{360})/0.007]} = 100,000/131.26156 = 761.84/month
\]

(b) \( A = 100,000 \quad k/m = 0.007 \quad mn = 15*12 = 180 \) months

\[
MP = \frac{100,000}{[(1-(1/1.007)^{180})/0.007]} = 100,000/102.15687 = 978.89/month
\]

**Additional Problems**

1. Mr. I. M. Thrifty expects to retire at age 65. Today is his fortieth birthday, and he has planned to make the following deposits in his savings account, which earns 6 percent compounded yearly. (All deposits will be on his birthday.)

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<tr>
<th>Age</th>
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<tr>
<td>41-54</td>
<td>1,000</td>
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<td>58-60</td>
<td>1,500</td>
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<td>62-64</td>
<td>2,000</td>
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Additionally, he expects his bank to increase the rate of interest to 8 percent after his fifty-sixth birthday. How much money will be in his bank account at age 65?

2. With reference to the problem above, how much can he withdraw from his bank account per year on his sixth-sixth through his eighty-eighth birthdays if he wants his account to be depleted by his eighty-eighth birthday?

3. Continuing with the problem above, how much could he withdraw from his bank account on his sixth-sixth through his eighty-eighth birthdays if he wants to leave $2,500 to be withdrawn on his ninety-fifth birthday?
4. A man arranges to repay a 6 percent, $1,000 loan in ten equal annual installments. After his second payment, he borrows an additional $1,000 with the following understanding: he is to pay nothing for the next two years and then to repay the balance on both loans in six equal annual payments. What will his annual payment be?

5. XYZ Loan Company has agreed to lend you $4,000 for eight years, if you agree to make one lump sum payment of $7,070.25 in eight years. What annual rate of interest is involved on his loan?

6. Mr. and Mrs. Debt recently borrowed $64,000. The interest on the loan is 12 percent. How much are the annual payments necessary to amortize this mortgage if the payment schedule is as follows: payment 2 is triple payment 1; payment 3 is twice payment 2; payment 4 is one-third of payment 3; and payment 5 and 6 are twice payment 4?

7. A bright finance student would like to buy a programmable calculator. She presently has $55 in her bank account. After carefully shopping for this calculator, she discovers that the calculator she wants costs $80. Assuming that calculators of this type are expected to increase in price at a 3 percent annual rate, at what rate of interest must she invest her $55 in order to be able to purchase this calculator eight years from now?

8. Ms. I Travel would like to have $2,000 to go on the Love Boat. If she presently has $1,400 in a savings account earning 5.25 percent, how long must she wait before she can make her trip?

Solutions To Additional Problems

Solutions to Computational Problems

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Since 6 percent will be earned until his fifty-sixth birthday: $21,015.07 \times (1.06)^2 = 23,612.53$.

Since the interest rate earned becomes 8 percent from his fifty-sixth to his sixty-fifth birthday: $23,612.53 \times (1.08)^9 = 47,201.56$.

\[ (b) \quad \$1,500 \left( \frac{(1.08)^3 - 1}{.08} \right) = \$4,869.60. \]
Since this money will continue earning interest at 8 percent until his sixty-fifth birthday: $4,869.60 \times (1.08)^5 = $7,155.04.

\[(c) \quad $2,000 \times \frac{(1.08)^5 - 1}{0.08} = $6,492.80\]

Again, this money will earn 8 percent for one additional period, therefore, $6,492.80 \times (1.08) = $7,012.22.

Answer: Amount available at age 65 will be $47,201.56 + $7,155.04 + $7,012.22 = $61,368.82.

2. This is basically a present value of an annuity problem. Mr. Thrifty has $61,368.82 available from which he wants to make 23 equal withdrawals. By the present value or annuity formula, we have

$$61,368.82 = P \left[ \frac{1 - (1 + 0.08)^{-23}}{0.08} \right]$$

Solving for $P$ yields: $5,917.31.

3. This problem becomes somewhat more complicated because at age 65, he must now leave enough money in the bank account (present value of an amount) for him to have $2,500 available at age 95. Therefore, the amount available for the annuity now becomes $61,368.82 minus the present value of $2,500 in 30 years.

$$61,368.82 - \frac{2,500}{(1.08)^{30}} = 61,368.82 - 248.44 = 61,120.38$$

$$61,120.38 = P \left[ \frac{1 - (1.08)^{-23}}{0.08} \right]$$

Solving for $P$, it now equals $5,893.36.

4. Initially, this is a present value ($1,000) of an annuity (payments) problem.

$$1,000 = P \left[ \frac{1 - 1.06^{-10}}{0.06} \right]; \quad P = $135.87.$$
After the first payment, the amount owed is $1,000 minus $75.87, or $924.13 ($60 of the payment was interest).

After the second payment, the outstanding balance is $924.13 - $80.42 = $843.71 ($55.45 of the second payment was interest).

Since $1,000 is borrowed, the balance owed is $1,843.71. Furthermore, since no payments will be made for two years, the amount owed in two years will be $1,843.71 \( (1.06)^2 \) = $2,071.59. Therefore, payments will be

\[
$2,071.59 = P \left[ 1 - \frac{1}{(1.06)^6} \right].
\]

Solving for \( P \), it equals $421.28.

5. To solve this problem we use the formula \( FV = PV(1 + k)^n \) and solve for the only unknown (\( k \)).

\[
7,970.25 = 4,000(1 + k)^8
\]
\[
1.9925625 = (1+k)^8
\]
\[
\sqrt[8]{1.9925625} = \sqrt[8]{(1+k)^8}
\]
\[
1.9925625^{1/8} = 1+k
\]
\[
1.09 = 1+k
\]
\[
k = .09.
\]

6. This is a present value problem with the addition of unequal payments. Using a little bit of algebra, this problem is easily solved:

\[
$64,000 = \frac{F}{1+k} + \frac{3F}{(1+k)^2} + \frac{6F}{(1+k)^3} + \frac{2F}{(1+k)^4} + \frac{4F}{(1+k)^5} + \frac{4F}{(1+k)^6}
\]
\[
$64,000 = \frac{F}{1.12} + \frac{3F}{(1.12)^2} + \frac{6F}{(1.12)^3} + \frac{2F}{(1.12)^4} + \frac{4F}{(1.12)^5} + \frac{4F}{(1.12)^6}
\]
Therefore,

\[ F = \frac{1}{1.12} + \frac{3}{(1.12)^2} + \frac{6}{(1.12)^3} + \frac{2}{(1.12)^4} + \frac{4}{(1.12)^5} + \frac{4}{(1.12)^6} \]

\[ $64,000 = F[0.8928571 + 2.39158164 + 4.2706754 + 1.2710356 + 2.2697096 + 2.0265272] \]

\[ $64,000 = F[13.122387]. \]

Therefore,

- Payment 1: \( F = $4,877.16 \)
- Payment 2: \( 3F = $14,631.48 \)
- Payment 3: \( 6F = $29,262.96 \)
- Payment 4: \( 2F = $9,754.32 \)
- Payment 5 & 6: \( 4F = $19,508.54 \).

7. The cost of his calculator in eight years will be \( 80(1.03)^8 = $101.34 \). Therefore, he must invest his $55 at a rate such that it will grow to $101.34.

\[ $55(1 + k)^8 = $101.34. \]

Using the same procedure as in problem 5, \( k \) is equal to .0794.

8. This problem involves the unknown period.

\[ n = \frac{\ln(T/P)}{\ln(1+k)} \]

\[ n = \frac{\ln\left(\frac{2,000}{1,400}\right)}{\ln(1.0525)} \]

\[ n = \frac{.3566749}{.0511683} \]

\[ n = 6.97 \text{ or approximately seven years.} \]

THEORETICAL PROBLEMS: THE TIME VALUE OF MONEY

1. Show that the nominal rate of interest \( k \) per year for an amount \( P \) to become \( F \) in \( n \) years in equal to

\[ k = m \left[ \left( \frac{F}{P} \right)^{1/mn} - 1 \right]. \]

2. Show that the number of years it will take an amount \( P \) to become \( F \) at a nominal rate of interest \( k \) when there are \( m \) compoundings per year is equal to
\[ n = \frac{\ln(F/P)}{m \ln \left(1 + k/m\right)} \quad 1 - \frac{P_0}{P} \cdot \frac{k}{m} \leq 0. \]

3. Show that the number of years in which an annuity of \(A_0\) per period will grow to \(A_1\), assuming the nominal interest rate \(k\) with \(m\) compoundings per year, is equal to

\[ n = \frac{\ln \left( \frac{k}{m} A_1 + 1 \right)}{\ln \left(1 + \frac{k}{m}\right)}. \]

4. Show that if equal withdrawals of \(P\) are made at the end of each period of length \(1/m\) year with an initial deposit of \(P_0\), the number of years required to exhaust the account will be equal to

\[ n = \frac{\ln \left[ 1 - \frac{P_0}{P} \cdot \frac{k}{m} \right]}{m \ln \left(1 + \frac{k}{m}\right)} \quad \text{for} \quad \frac{P_0}{P} \cdot \frac{k}{m} < 1 \]

\[ = \infty \quad \text{for} \quad \frac{P_0}{P} \cdot \frac{k}{m} \geq 1. \]

5. Show that the relationship between the annual effective and nominal rates of interest is independent for the amount of money deposited and number of years left in an account. That is, show that the relationship between \(i\) and \(k\) does not depend on \(A\) (the amount deposited) or \(n\).

6. Show that in the case of continuous compounding the effective rate of interest \(i\) and the nominal rate of interest \(k\) are related as follows:

\[ i = e^k - 1 \]

and

\[ k = \ln(1 + i). \]

7. Assume that a deposit of \(A\) was made today. If the nominal rates of interest are \(k_1, k_2, \ldots, k_m\) during periods 1, 2, ..., \(n\) show that the future value of \(A\) at the end of year \(n\), assuming \(m\) compoundings per year, is
and assuming continuous compounding in each year is

\[ Ae^{k_1+k_2+\ldots+kn} \]

8. Using interest rates and periods as in the previous question, show that the present value of an amount to be received at the end of \( n \) years is

\[ A\left(1 + \frac{k_1}{m}\right)^{-m} \left(1 + \frac{k_2}{m}\right)^{-m} \ldots \left(1 + \frac{k_n}{m}\right)^{-m} \]

for \( m \) compoundings per year and

\[ Ae^{-(k_1+k_2+\ldots+kn)} \]

for the case of continuous compounding.

9. Show that for the continuous compounding case

\[ FVA^c = (PVA^c)e^{nk} \]

and

\[ (PVA^c) = (FVA^c)e^{-nk}. \]

10. Show that the nominal rate of interest \( k \) that is required for an amount \( P \) to be worth a value of \( F \) in \( n \) years, assuming continuous compounding, is equal to

\[ k = \frac{1}{n} \times \ln\left(\frac{F}{P}\right). \]

11. Show that the number of years required for an amount \( P \) to be worth a value of \( F \) compounded continuously is equal to

\[ n = \frac{1}{k} \times \ln\left(\frac{F}{P}\right). \]

12. Show that the number of years in which an annuity of \( A_0 \) will grow to \( A_1 \), assuming a nominal rate of interest \( k \), compounding continuously is

\[ n = \frac{1}{k} \times \ln\left[1 + \frac{A_1}{A_0}(e^k - 1)\right]. \]

13. Show that the number of years required for equal withdrawals of \( A_l \) at the end of each year to exhaust an initial deposit of \( A_0 \) is
\[ - \frac{1}{k} \ln \left[ 1 - \frac{A_0}{A_1} (e^k - 1) \right] \quad \text{for} \quad 1 - \frac{A_0}{A_1} (e^k - 1) > 0 \]

\[ n = \infty \quad \text{for} \quad 1 - \frac{A_0}{A_1} (e^k - 1) \leq 0 \]

**Solutions to Theoretical Problems**

1. To show this, we begin with the definition of the future value of an amount for the general compounding case [equation (2.16)].

\[ F = P \left( 1 + \frac{k}{m} \right)^{mn} \]

Using the rules for exponents, we solve for \( k \) as

\[ \left( \frac{F}{P} \right)^{1/mn} = 1 + \frac{k}{m} \]

or

\[ \frac{k}{m} = \left( \frac{F}{P} \right)^{1/mn} - 1 \]

or

\[ k = m \left( \left( \frac{F}{P} \right)^{1/mn} - 1 \right) \]

2. Starting once again with equation (2.16) but this time solving for \( n \), we have

\[ F = P \left( 1 + \frac{k}{m} \right)^{mn} \]

Taking the natural log of both sides, we have

\[ \ln F = \ln P + mn \ln \left( 1 + \frac{k}{m} \right) \]

Solving for \( n \) yields

\[ n = \frac{\ln F - \ln P}{mn \ln \left( 1 + \frac{k}{m} \right)} = \frac{\ln(F / P)}{m \ln \left( 1 + \frac{k}{m} \right)} \]
3. We use basically the same procedure as in the previous problem but we start with the
definition of the future value of an annuity.

\[ A_n = A_0 \left( \frac{\left(1 + \frac{k}{m}\right)^{mn} - 1}{\frac{k}{m}} \right) \]

\[ \frac{A_n}{A_0} = \frac{\left(1 + \frac{k}{m}\right)^{mn} - 1}{\frac{k}{m}} \]

\[ \frac{k}{m} A_n + 1 = \left(1 + \frac{k}{m}\right)^{mn} \]

Taking the log of both sides and solving for \( n \), we have

\[ \ln \left( \frac{k}{m} A_n + 1 \right) = mn \ln \left(1 + \frac{k}{m}\right) \]

or

\[ n = \frac{\ln \left( \frac{k}{m} A_n + 1 \right)}{m \ln \left(1 + \frac{k}{m}\right)} \]

4. Again we apply the same procedure, this time starting with the present value of an
annuity formula. Let \( P \) represent the equal withdrawals and \( P_0 \) the initial amount.
Then

\[ P_0 = P \left(1 - \left(1 + \frac{k}{m}\right)^{-mn}\right) \frac{k}{m} \]

Rearranging terms, we have

\[ 1 \frac{m P}{k P_0} = \left(1 + \frac{k}{m}\right)^{-mn} \]
Taking logs and solving for \( n \), we get

\[
\ln \left[ 1 - \frac{P m}{P_0 k} \right] = -nm\ln\left(1 + \frac{k}{m}\right)
\]

\[
n = -\frac{\ln \left[ 1 - \frac{P m}{P_0 k} \right]}{m\ln\left(1 + \frac{k}{m}\right)} \text{ for } 1 - \frac{P m}{P_0 k} > 0
\]

\[
= \infty \text{ otherwise.}
\]

5. To show this, let \( k \) be the nominal rate of interest and \( i \) the annual effective rate of interest. The future value, \( F \), of any amount \( A \) can be written as

\[
A\left(1 + \frac{k}{m}\right)^{nm} = F \text{ for the nominal interest rate; and}
\]

\[
A(1 + i)^n = F \text{ for the effective annual rate.}
\]

Equating these, we have

\[
A\left(1 + \frac{k}{m}\right)^{nm} = A(1 + i)^n.
\]

Solving this expression for \( k \) yields

\[
1 + \frac{k}{m} = (1 + i)^{1/m}
\]

or

\[
k = m[(1 + i)^{1/m} - 1].
\]

Since this expression for \( k \) contains no terms with \( A \) or \( n \) in it, the relationship must be independent of these variables.

6. To show this, let us start with the definition of future value as in the previous problem, but now assuming continuous compounding:

\[
Ae^{kn} = F \text{ for the continuous compounding case; and }
\]

\[
A(1 + i)^n = F \text{ to determine the annual effective rate.}
\]
Equating the two expressions and canceling the $A$'s, we have

$$e^{kn} = (1 + i)^n.$$ 

Taking logs and solving for $k$, we have

$$k = \frac{n}{n} \ln(1 + i) = \ln(1 + i).$$

Instead of solving for $k$, we could have solved for $i$ as

$$(e^{kn})^{1/n} = 1 + i$$

$$i = e^k - 1.$$ 

7. Note that each period here represents one year. The future value of an amount at the end of one year, $F_1$, can be written as

$$F_1 = A\left(1 + \frac{k_1}{m}\right)^m.$$ 

The future value of the amount at the end of the second year will be the future value of $F_1$ after one year of earning interest at a rate $k_2$.

Thus,

$$F_2 = F_1 \left(1 + \frac{k_2}{m}\right)^m = A\left(1 + \frac{k_1}{m}\right)^m \left(1 + \frac{k_2}{m}\right)^m.$$ 

Carrying this out for $n$ years, we would get

$$F = A\left(1 + \frac{k_1}{m}\right)^m \left(1 + \frac{k_2}{m}\right)^m \cdots \left(1 + \frac{k}{m}\right)^m.$$ 

Perform the same steps to show this in the case of continuous compounding. That is,

$$F_1 = Ae^{k_1}.$$ 

$$F_2 = F_1 e^{k_2} = Ae^{k_1} e^{k_2} = Ae^{k_1 + k_2}.$$ 

For $n$ periods, the future value, $F$, is

$$F = Ae^{k_1 + k_2 + \cdots + k_n}.$$
8. The present value of an amount $A$ to be received one year from now with an interest rate of $k_1$ is given by

$$PV^1 = A \left(1 + \frac{k_1}{m}\right)^{-m}.$$ 

The present value of this amount if it were received after another year (two years from time 0) at interest rate $k_2$ for year 2 would be

$$PV^2 = PV^1 \left(1 + \frac{k_2}{m}\right)^{-m}.$$ 

Substituting for $PV^1$, we have

$$PV^2 = A \left(1 + \frac{k_1}{m}\right)^{-m} \left(1 + \frac{k_2}{m}\right)^{-m}.$$ 

Carrying this out for $n$ years would yield

$$PV = A \left(1 + \frac{k_1}{m}\right)^{-m} \left(1 + \frac{k_2}{m}\right)^{-m} \cdots \left(1 + \frac{k_n}{m}\right)^{-m}.$$ 

For the case of continuous compounding,

$$PV^1 = Ae^{-k_1}; \text{ and}$$

$$PV^2 = PV^1 e^{-k_2} = Ae^{k_1} e^{-k_2} = Ae^{-(k_1 + k_2)}.$$ 

Extending this for $n$ years, we have

$$PV = Ae^{-(k_1 + k_2 + \cdots + k_n)}.$$ 

9. From equation (2.22),

$$FVA^c = \frac{P(e^{nk} - 1)}{e^k - 1}.$$ 

Multiplying both sides by $e^{-nk}$, we have

$$FVA^c e^{-nk} = \frac{P(e^{nk} e^{-nk} - e^{-nk})}{e^k - 1}.$$
\[ PVA^e = \frac{P(1 - e^{-nk})}{e^k - 1} \]

\[ FVA^e e^{-nk} = PVA^e, \]

or by dividing through by \( e^{nk} \).

\[ PVA^e e^{-nk} = FVA^e. \]

10. From equation (2.20), the present value of an amount to be received in the future is given by

\[ P = Fe^{-kn} \]

or

\[ e^{kn} = F / P. \]

Taking the log of both sides and solving for \( k \), we have

\[ kn \ln e = \ln(F / P) \]

or

\[ k = \frac{1}{n} \ln(F / P) \]

11. From the above problem, we simply solve for \( n \) instead of \( k \). That is,
12. The future values of an annuity of $A_0$ dollars (call its value $A_f$ is given by equation (2.22):

$$A_f = \frac{A_0 \left(e^{nk} - 1\right)}{e^k - 1}.$$ 

Rearranging terms, we have

$$e^{nk} = \frac{A_f}{A_0} (e^k - 1) + 1.$$ 

Taking the log of both sides and solving for $n$ yields

$$nk \ln e = \ln \left[ \frac{A_f}{A_0} (e^k - 1) + 1 \right]$$

or

$$n = \frac{1}{k} \ln \left[ 1 + \frac{A_f}{A_0} (e^k - 1) \right].$$

13. The present value of an annuity of $A_f$ dollars is given in equation (2.23) as

$$A_0 = \frac{A_f \left(1 - e^{-nk}\right)}{e^k - 1}.$$ 

Rearranging terms, we have

$$1 - e^{-nk} = \frac{A_0}{A_f} (e^k - 1)$$

or

$$e^{-nk} = 1 - \frac{A_0}{A_f} (e^k - 1).$$
Taking the log of both sides and solving for $n$, we have

$$-nk = \ln \left[ 1 - \frac{A_0}{A_1} (e^k - 1) \right]$$

or

$$n = \frac{-1}{k} \ln \left[ 1 - \frac{A_0}{A_1} (e^k - 1) \right]$$

Note that if the term

$$1 - \frac{A_0}{A_1} (e^k - 1)$$

is less than or equal to zero, the log function is undefined; hence, this expression for $n$ is only valid if the above term is greater than zero. If the term is less than or equal to zero, it implies that withdrawals of $A_1$ can never exhaust the initial deposit.