ground water

Nonlinear Flow in Karst Formations

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Abstract

The variation of effective hydraulic conductivity as a function of specific discharge in several 0.2-m and 0.3-m cubes of Key Largo Limestone was investigated. The experimental results closely match the Forchheimer equation. Defining the pore-size length scale in terms of Forchheimer parameters, it is demonstrated that significant deviations from Darcian flow will occur when the Reynolds number exceeds 0.11. A particular threshold model previously proposed for use in karstic formations does not show strong agreement with the data near the onset of nonlinear flow.

Introduction

Characterization of nonlinear flow in porous media has been a topic of research for many years (e.g., Dudgeon 1966; Arbhabhirama and Dinoy 1973; Venkataraman and Rao 1998), and a summary of many of the nonlinear theories can found in Bear (1972). Based on experimental results in a variety of porous media, it is generally agreed that at low Reynolds numbers the head gradient is linearly proportional to the specific discharge and Darcy's law is valid, whereas at high Reynolds numbers the head gradient depends nonlinearly on the specific discharge and Darcy's law is not valid (e.g., Freeze and Cherry 1979; Bear 1972). In this context, the Reynolds number is generally defined as

$$\mathrm{Re} = \frac{\rho V d}{\mu} \tag{1}$$

where ρ and μ are the density and dynamic viscosity of the fluid, respectively, V is the specific discharge, and d is the characteristic pore size. To accommodate nonlinearities in the relationship between head gradient and specific discharge, the following Darcian-like relationship can be used:

$$V_i = -K_{\mathrm{eff},i} \frac{\partial h}{\partial x_i} \tag{2}$$

where V_i and $K_{\mathrm{eff},i}$ are the specific discharge and effective hydraulic conductivity, respectively, in the x_i direction, and h is the total head. The effective hydraulic conductivity in any flow direction is therefore defined as the ratio of the specific discharge to the head gradient in that direction, regardless of the flow regime. In conventional practice, it is common to define a critical Reynolds number, $\mathrm{Re}_{\mathrm{crit}}$, where the flow is represented as Darcian or linear (i.e., $K_{\mathrm{eff},i}$ = constant) when $\mathrm{Re} \leq \mathrm{Re}_{\mathrm{crit}}$ and non-Darcian or nonlinear when $\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$. In unconsolidated formations, where d is taken as the mean grain size, typical values of $\mathrm{Re}_{\mathrm{crit}}$ are in the range of 1 to 10, and nonlinear flows are taken to occur when $\mathrm{Re} > 10$. Nonlinear flows can be laminar or turbulent depending on the Reynolds number, with transitional flow (from laminar to turbulent) occurring when $10 < \mathrm{Re} < 100$ and fully turbulent flow when $\mathrm{Re} > 100$ (Bear 1972).

A general constitutive equation that can be used in lieu of the Darcy equation to describe both linear and nonlinear flow in porous media is the Forchheimer equation (Forchheimer 1930), given by

$$\frac{\partial h}{\partial x_i} = aV_i + bV_i^2 \tag{3}$$

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where a and b are constants. Equation 3 is grounded in the fluid mechanics of flow in porous media, as shown by Ahmed and Sunada (1969), who used the Navier-Stokes equations to show that a and b are related to the properties of the fluid and porous medium by

$$a = \frac{\mu}{\rho g k} \text{ and } b = \frac{1}{g\sqrt{ck}} \tag{4}$$

where k is the intrinsic permeability (which is independent of flow regime) and c is the proportionality factor defined by the relation

$$k = cd^2 \tag{5}$$

and d is a characteristic pore dimension. The relationships in Equation 4 were also derived by Ward (1966) using dimensional analysis, and the combination of Equations 3 and 4 has been validated by Venkataraman and Rao (1998) using the results from numerous investigations in unconsolidated porous media.

There is a paucity of data and supporting models describing nonlinear flows in karstic formations. Shoemaker et al. (2008) developed the Conduit Flow Process (CFP) module for MODFLOW-2005 (Harbaugh 2005) to simulate nonlinear flow in karstic aquifers, and an option within the CFP module is to simulate flow between cells with continuous preferential flow layers, where flow can transition between linear and nonlinear conditions. Flow in these karstic layers is represented as Darcian according to the relation

$$V_i = -K_{\text{eff},\,i}\frac{\Delta h}{\Delta x_i} \tag{6}$$

where $\Delta h/\Delta x_i$ is the hydraulic head gradient in the i direction. In any given flow direction, the effective hydraulic conductivity, K_{eff} (the i has been dropped for notational convenience), is given in normalized form as

$$\frac{K_{\text{eff}}}{K_0^D} = \begin{cases} \sqrt{\frac{\text{Re}_{\text{crit}}}{\text{Re}}} & \text{Re} > \text{Re}_{\text{crit}} \\ 1 & \text{Re} \leq \text{Re}_{\text{crit}} \end{cases} \tag{7}$$

where K_0^D is the hydraulic conductivity under Darcian (linear) flow conditions and Re_{crit} is the limiting Reynolds number for approximating the hydraulic conductivity as K_0^D. In contrast to Equation 7, the Forchheimer model for estimating the effective hydraulic conductivity under all flow regimes can be obtained by combining Equations 3 and 6 and is given by

$$\frac{K_{\text{eff}}}{K_0^F} = \frac{1}{1+\text{Re}} \tag{8}$$

where K_0^F and Re are derived from the Forchheimer parameters, a and b, according to the relations

$$K_0^F = \frac{1}{a} \text{ and } \text{Re} = V\left(\frac{b}{a}\right) \tag{9}$$

In Equations 7 and 8, K_0^D and K_0^F both represent the hydraulic conductivity when nonlinear effects are negligible. However, these parameters are not directly comparable since K_0^D is the (constant) hydraulic conductivity when $\text{Re} \leq \text{Re}_{\text{crit}}$ in the Shoemaker threshold model, and K_0^F is the asymptotic hydraulic conductivity as $\text{Re} \rightarrow 0$ in the Forchheimer model.

The objective of this paper is to analyze permeameter data collected from a karst formation to assess the relative accuracy of using Equations 7 and 8 to describe the nonlinear behavior of the effective hydraulic conductivity, and to identify critical flow conditions beyond which nonlinear effects must be taken into account in karstic aquifers.

Materials and Methods

The methods used to collect the data in this study have been previously reported by DiFrenna et al. (2007) and are briefly summarized here. Seven cubes 0.2 m on a side and six cubes 0.3 m on a side were cut from a larger block of Key Largo Limestone. The orientation of the large block was maintained during cutting so that the vertical axis of each cube was known. Numerous solution holes ranging in size from < 1 cm to about 4 cm long were observed in the limestone cubes. The hydraulic conductivity of each cube axis was determined using a Plexiglas permeameter. To prevent preferential flow around the cube, the outer faces of the cube were first wrapped with plastic wrap, then with a sheet of 0.635-cm-thick closed-cell neoprene rubber, followed by 0.636-cm-thick aluminum plates that were tightened with nylon straps and a ratcheting mechanism. All seams of the Plexiglas chamber were sealed with silicone, and the integrity of the assembly was verified during testing by the absence of leaks and after testing by confirming that the neoprene was dry and there were no holes in the plastic wrap. The volumes of water passing through the cubes at timed intervals for various static head differences across the cube were measured, and the results of the experiments were reported as specific discharge vs. head gradient. The minimum head gradient imposed was 0.08 m/m and the maximum head gradient imposed was 3 m/m. It was verified that when the head gradient was zero, there was no flow through the cube. At least eight static head gradients were used on each cube axis, resulting in more than 300 measurements. Permeameter experiments were performed on 13 cubes, with each experiment identified by the rubric *c-sf*, where *c* is the cube number (= 1 to 7), *s* is the cube size (2 = 0.2 m; 3 = 0.3 m), and *f* is the flow direction (a = vertical; b = horizontal-1; c = horizontal-2).

Data Analysis

For each experiment, model parameters that give the best fit, in terms of least squares, to the data were determined using a computer program developed specifically for this task. For the Forchheimer equation, the objective function, $\Delta^2(a, b)$, was

$$\Delta^2(a,b) = \sum_{i=1}^{N} \left(av_i + bv_i^2 - s_i\right)^2 \tag{10}$$

where N is the number of measurements, and v_i and s_i are the specific discharge and head gradient, respectively, in measurement i. Inclusion of $v_i = 0$ when $s_i = 0$ as a measurement point was confirmed by a lack of observed discharge when the head gradient was zero. There were no measurement points between (0,0) and (0.24 m/d, 0.08 m/m), and the lack of data in this range could bias the results presented here. For the data set corresponding to each experiment, the values of a and b that yield the least value of $\Delta^2(a, b)$ in Equation 10 were found. For the Shoemaker et al. (2008) equation, the specific discharge, V_i, can be expressed in the form

$$V_i = \begin{cases} K_0^D \sqrt{\frac{\mathrm{Re_{crit}}}{\mathrm{Re}}} \frac{\Delta h}{\Delta x_i}, & \mathrm{Re} > \mathrm{Re_{crit}} \\ K_0^D \frac{\Delta h}{\Delta x_i}, & \mathrm{Re} \leq \mathrm{Re_{crit}} \end{cases} \tag{11}$$

and the objective function, $\Delta^2(K_0^D, \mathrm{Re_{crit}})$, for determining the optimal values of K_0^D and $\mathrm{Re_{crit}}$ was

$$\Delta^2\left(K_0^D, \mathrm{Re_{crit}}\right) = \sum_{i=1}^{N} \left\{ \left[K_0^D \sqrt{\frac{\mathrm{Re_{crit}}}{\mathrm{Re}_i}} s_i - v_i \right]^2_{\text{if } \mathrm{Re}_i > \mathrm{Re_{crit}}} + \left[K_0^D s_i - v_i\right]^2_{\text{if } \mathrm{Re}_i \leq \mathrm{Re_{crit}}} \right\} \tag{12}$$

where $\mathrm{Re}_i = v_i d/\nu$, and it was assumed that $d = 0.7$ cm and $\nu = 10^{-6}$ m^2/s; however, the assumed values of d and ν had no influence on the optimal solution because these values cancel out on the right-hand side of Equation 12. For the data set corresponding to each experiment, the values of K_0^D and $\mathrm{Re_{crit}}$ that yield the least value of $\Delta^2(K_0^D, \mathrm{Re_{crit}})$ in Equation 12 were found.

The optimum parameter sets for both the Forchheimer and Shoemaker models are shown in Table 1, from which it is apparent that the values of K_0^D and K_0^F are of similar magnitude. In almost all experiments, the relationship between the head gradient and specific discharge was sufficiently nonlinear that $\mathrm{Re_{crit}}$ for the threshold (Shoemaker) model was within the range of Re for the experimental data; however, in a few experiments (2-3b, 2-3c, 4-3c), the relationship between head gradient and specific discharge was sufficiently linear over the range of the experimental data that $\mathrm{Re_{crit}}$ was indeterminate. The characteristic pore size, d, of the karst matrix is the key variable affecting the initiation of nonlinear flow in a porous medium and, according to Equations 4 and 5, is related to the Forchheimer parameters by the relation

$$d = \nu\left(\frac{b}{a}\right) \tag{13}$$

This relation was used to obtain values of d for each of the experimental cubes of Key Largo limestone, and the median value of d was found to be 0.7 cm, which is consistent with the pore sizes observed in the sides of the cubes (DiFrenna et al. 2007). The range of d in the experimental results was 0.2 to 10.3 cm.

Table 1
Parameters for Nonlinear Flow Models

	Forchheimer Model		Shoemaker Model	
Experiment	K_0^F **(= 1/*a*) (m/d)**	***a/b* (m/s)**	K_0^D **(m/d)**	$\mathbf{Re_{crit}}$
1-2a	17.6	6.6	14.2	2.0
1-2b	9.4	19.6	7.8	4.2
1-2c	3.6	10.1	3.1	1.6
2-2a	1.0	2.6	0.8	0.5
2-2b	1.2	3.1	1.0	0.5
2-2c	0.8	0.8	0.6	0.3
3-2a	3.0	9.0	2.6	1.5
3-2b	3.2	1.8	2.2	0.8
3-2c	0.9	7.3	0.8	0.7
4-2a	2.2	24.3	2.1	1.6
4-2b	4.3	21.9	3.9	2.6
4-2c	6.5	5.2	4.9	1.8
5-2a	15.9	6.8	13.5	1.9
5-2b	5.8	20.2	5.1	3.2
5-2c	3.5	23.3	3.2	2.5
6-2a	52.3	42.6	44.1	11.1
6-2b	79.9	26.8	64.8	8.1
6-2c	40.0	45.0	31.0	13.4
7-2a	12.4	43.2	10.8	7.0
7-2b	29.9	25.3	23.3	7.8
7-2c	20.4	20.3	15.3	6.7
1-3a	3.2	7.9	2.7	1.6
1-3c	0.6	2.5	0.5	0.3
2-3b	0.2	[1]	0.2	[1]
2-3c	0.3	[1]	0.3	[1]
3-3a	0.7	30.6	0.7	0.5
3-3b	1.5	11.1	1.4	1.0
4-3a	2.2	20.7	2.1	1.6
4-3b	3.5	9.4	3.0	1.7
4-3c	1.8	[1]	1.8	[1]
5-3a	15.4	4.7	9.1	2.6
5-3b	278.6	16.2	216.6	4.0
5-3c	320.4	12.7	155.4	7.3
6-3a	0.9	6.7	0.8	0.6
6-3b	38.7	22.3	31.7	6.5
6-3c	43.6	8.9	58.4	1.0

[1] $\mathrm{Re_{crit}}$ indeterminate, experiment in Darcian (linear) range.

Experiments on each cube provided more than 300 measurements of specific discharge vs. head gradient. Because the effective hydraulic conductivity is equal to the ratio of the specific discharge to the head gradient, the collected data for each cube were converted to effective hydraulic conductivity, K_{eff}, vs. specific discharge, V. For comparison with the Forchheimer equation, best-fit values of a and b were found, K_{eff} was normalized with 1/a to give K_{eff}/K_0^F, V was normalized with a/b to give Re, and the results from all experiments are compared with the Forchheimer equation (Equation 8) in Figure 1. The Forchheimer equation provides good agreement with the data for all values of Re. For comparison with the Shoemaker equation, best-fit values of K_0^D, and $\mathrm{Re}_{\mathrm{crit}}$ were found, K_{eff} was normalized with K_0^D, and Re was normalized with $\mathrm{Re}_{\mathrm{crit}}$; the results are compared with the Shoemaker equation (Equation 7) in Figure 2. The Shoemaker equation shows good agreement with the experimental data for $\mathrm{Re}/\mathrm{Re}_{\mathrm{crit}} > 1$, but poor agreement for $\mathrm{Re}/\mathrm{Re}_{\mathrm{crit}} \leq 1$, and the assumption of an abrupt transition at $\mathrm{Re}_{\mathrm{crit}}$ does not fit the data well. The median value of $\mathrm{Re}_{\mathrm{crit}}$ was found to be 1.8, and the range of $\mathrm{Re}_{\mathrm{crit}}$ was 0.3 to 13.4. For $\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$, both the Shoemaker and Forchheimer equations provide good descriptions of the behavior of the effective hydraulic conductivity, and this result could partially explain the anomalous results reported by Kuniansky et al. (2008), who found good performance of the Shoemaker model for a single cube where $\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$ for 17 of the 24 measurements considered in their analysis. The range of flow conditions covered in Figures 1 and 2 is representative of practical flow conditions in karstic formations. A curious feature of the fit to the Shoemaker equation is that $K_{\mathrm{eff}}/K_0^D > 1$ when $\mathrm{Re}/\mathrm{Re}_{\mathrm{crit}} \rightarrow 0$. The reason for this is that the specific discharge vs. head gradient curve is actually concave downward (due to nonlinearity) and so the best-fit K_0^D to the assumed Darcian part of the curve yields an average K_0^D that is less than K_{eff} as $\mathrm{Re} \rightarrow 0$. This is illustrated in Figure 3 for the Darcian range of measurements in Experiment 6-2c. Although this nonlinearity was observed by DiFrenna et al. (2007) at $\mathrm{Re} < 1$, the data points fell within the 95% confidence interval of a linear best-fit line; therefore, a linear approximation could be used up to $\mathrm{Re} \approx 1$ (based on $d = 1$ cm). However, DiFrenna et al. (2007) explicitly recognized the nonlinearities in the data at $\mathrm{Re} < 1$ by stating that the average hydraulic conductivity (K_0^D) over the assumed Darcian (linear) range would be less than the effective hydraulic conductivity (K_{eff}) at lower values of the Reynolds number, a condition that is reflected in Figures 2 and 3 and is the stimulus for this paper, which proposes an alternate model that accounts for the observed systematic nonlinearity in the data at low Reynolds numbers.

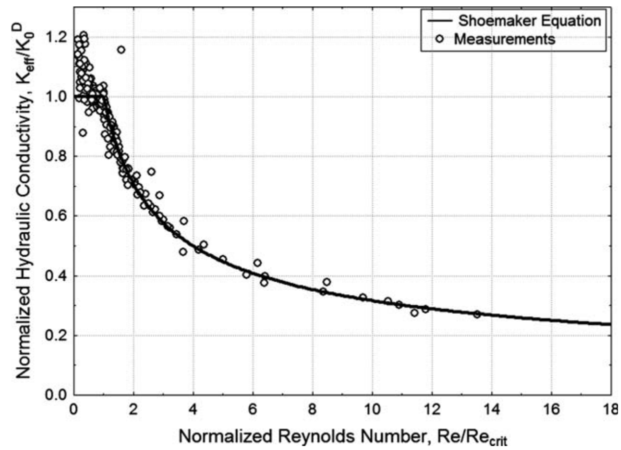

Figure 2. Shoemaker et al. (2008) equation and measurements.

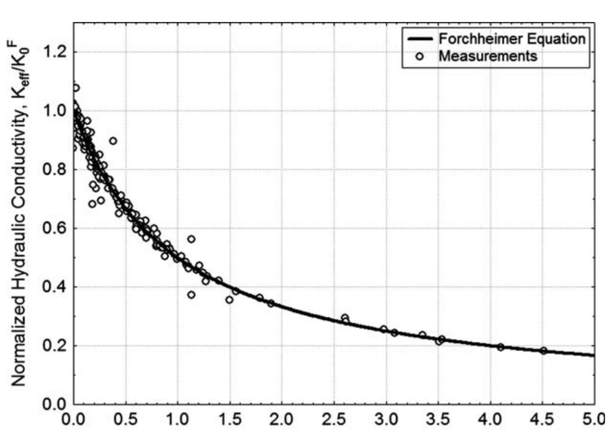

Figure 1. Forchheimer equation and measurements.

Discussion

The primary result reported in this paper is that the effective hydraulic conductivity of the Key Largo limestone cubes tested in this study is closely described by the Forchheimer equation and can be expressed in the form

$$\frac{K_{\mathrm{eff}}}{K_0^F} = \frac{1}{1 + \mathrm{Re}} \tag{14}$$

Apparent from Equation 14 is that $K_{\mathrm{eff}}/K_0^F = 0.9$ when $\mathrm{Re} = 0.11$, and hence $\mathrm{Re} = 0.11$ can be defined as the condition for significant deviation from Darcian flow. Taking $d = 0.7$ cm (the median value in the experiments) and $\nu = 10^{-6}$ m²/s with $\mathrm{Re} = 0.11$ gives $V \approx 1$ m/d, and so nonlinear flow in this karstic formation can be expected when the specific discharge exceeds about

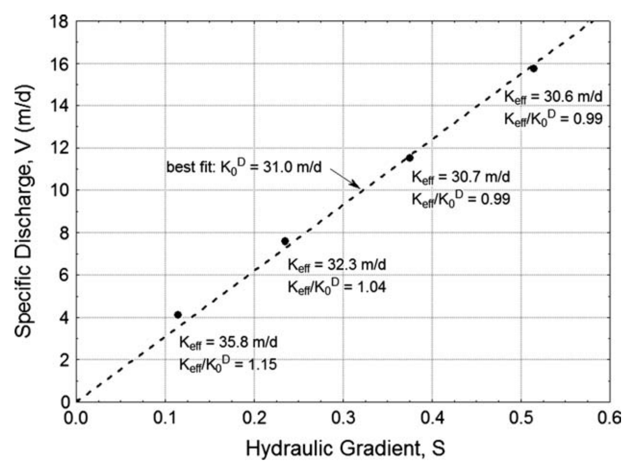

Figure 3. Darcian measurements in experiment 6-2c.

1 m/d. In general, this limiting velocity, V_{lim}, is given by $V_{\mathrm{lim}} = 0.11\ v/d$. A corollary to this result is that significant deviation from Darcian flow in a karstic formation will occur at specific discharges at least an order of magnitude lower than in unconsolidated formations, because the d values found here are at least an order of magnitude larger than the values typical of unconsolidated formations.

At first glance, it might seem anomalous that deviation from Darcian flow is predicted to occur at Re $= 0.11$, whereas in conventional practice it is generally accepted that significant deviation from Darcian flow does not occur until Re > 1 to 10. However, the conventional criterion is based on the specific definition of the length scale, d, as the mean grain size of the unconsolidated porous medium (Bear 1972). If another definition of d is used, then the limiting Re for Darcian flow is different from the conventional criterion of Re > 1 to 10. In the case of a karstic formation, identifying a mean grain size is not meaningful, and the alternative definition of d that is implicit in the Forchheimer formulation is

$$d = \frac{b}{a} v \qquad (15)$$

where a and b are the Forchheimer parameters and v is the kinematic viscosity of water. Because d is defined by Equation 15 rather than a mean grain size, the conventional criterion of Re > 1 to 10 for the initiation of non-Darcian flow is not a relevant benchmark for the results reported here, where the limiting condition is Re $= 0.11$. Venkataraman and Rao (1998) used the same definition of d as given by Equation 15 and identified Darcian flow as occurring when Re < 0.1 (approximate), nonlinear Darcian flow when $0.1 <$ Re < 10, and fully turbulent flow when Re > 10. Based on these classifications, it is apparent from Figure 1 that flow conditions in the current experiments were all under linear or nonlinear Darcian flow conditions, and none were under fully turbulent flow conditions.

A possible reason why the Shoemaker threshold model does not match the data well could be related to the assumed functional form of the model and the requirement that $\mathrm{Re}_{\mathrm{crit}}$ and K_0^D be such that the data best matches the assumed functional relationship between the K_{eff} and V. The assumed functional form in the Shoemaker threshold model requires that the flow suddenly transition from Darcian (linear) flow, where $K_{\mathrm{eff}} =$ constant, to a transitional flow condition where $\mathrm{K}_{\mathrm{eff}} \propto V^{-0.5}$, whereas, in reality, $\mathrm{K}_{\mathrm{eff}}$ is expected to vary gradually from being a constant to being proportional to V^{-1} when the flow becomes fully turbulent (Venkataraman and Rao 1998). A possible solution to improve the performance of the threshold model is to use an alternative transition flow function that better describes the flow conditions at the onset of nonlinear flow, which would most likely result in a lower $\mathrm{Re}_{\mathrm{crit}}$ and a better match to the experimental data at low Reynolds numbers.

Conclusions

Both the classical Forchheimer equation and a threshold model proposed by Shoemaker et al. (2008) for use in preferential flow layers of karst formations were compared with the experimental results. This comparison showed that the experimental results closely matched the Forchheimer equation, whereas the threshold model does not adequately identify the critical Reynolds number at which the flow becomes significantly nonlinear. Using a pore-size length scale derived from the Forchheimer parameters, it is demonstrated that the effective hydraulic conductivity is less than 90% of the Darcian (linear) flow hydraulic conductivity when the Reynolds number exceeds 0.11.

Practical applications of these results include the development of a Forchheimer option in the CFP module for MODFLOW to simulate nonlinear flow in karstic aquifers as well as a renewed interest in the application of analytical results describing Forchheimer flow in the vicinity of large water-supply wells (e.g., Mathias et al. 2008), particularly in karstic formations. In support of these applications, further research is recommended to investigate the upscaling of these results to larger sample blocks.

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