#5 (a) div(x, y) was defined to be the largest integer n such that $x \geqslant ny$. When $y=0$, div(x, y) is undefined because there is no largest integer n such that $x \geqslant n.0$. So we will amend the definition as follows:

$$\operatorname{div}(x,y) = \begin{cases} 0 & \text{if } y=0 \\ \text{largest } n \text{ such that } x \geq ny & \text{if } y \neq 0. \end{cases}$$

Now
$$\operatorname{div}(0,y) = 0$$
$$\operatorname{div}(x+1,y) = \begin{cases} \operatorname{div}(x,y) & \text{if } x+1 < y(1+\operatorname{div}(x,y)) \\ \operatorname{div}(x,y)+1 & \text{if } x+1 = y(1+\operatorname{div}(x,y)) \end{cases}$$

$\therefore$
$$\operatorname{div}(0,y) = 0$$
$$\operatorname{div}(x+1,y) = \operatorname{div}(x,y) + \operatorname{equals}(x+1,\ y+(y.\operatorname{div}(x,y)))$$

Now this is almost okay except the recursion is being done on the first coordinate. In primitive recursion, we have to do it on the last coordinate.

$$\operatorname{div}(x,y) = f(y,x) = f(p_2(x,y),\ p_1(x,y))$$

where
$$f(x,0) = z(x)$$
$$\begin{aligned} f(x,y+1) &= \operatorname{div}(y+1,x) \\ &= \operatorname{div}(y,x) + \operatorname{equals}(y+1,\ x+x.\operatorname{div}(y,x)) \\ &= f(x,y) + \operatorname{equals}(y+1,\ x+x.f(x,y)) \\ &= h(x,y,f(x,y)) \end{aligned}$$

where
$$h(x,y,z) = \operatorname{add}(p_3(x,y,z),\ \operatorname{equals}(s(p_2(x,y,z)),\ \operatorname{add}(p_1(x,y,z),\ \operatorname{mult}(p_1(x,y,z),\ p_3(x,y,z))).$$

$\therefore$ div(x, y) is primitive recursive.

5 (b) $rem(x,y) = x \dot{-} y.div(x,y)$
$= monus(p_1(x,y),\ mult(p_2(x,y),\ div(x,y)))$

(c) $max(x,y) = x + y \dot{-} x$
$= add(p_1(x,y),\ monus(p_2(x,y),\ p_1(x,y)))$

(d) $min(x,y) = x \dot{-} (x \dot{-} y)$
$= monus(p_1(x,y),\ monus(p_1(x,y),\ p_2(x,y)))$

(e) $gr(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$
$= 1 \dot{-} equal(x \dot{-} y, 0)$

$= monus(c_1(p_1(x,y)),\ equal(monus(x,y),\ z(p_1(x,y))))$

(f) $f(x) = \lfloor \sqrt{x} \rfloor = (\mu y)((y+1)^2 > x)$
$= (\mu y)[g(x,y) = 0]$,
where $g(x,y) = 1 \dot{-} gr((y+1)^2, x) =$
$monus(c_1(p_1(x,y)),\ gr(mult(s(p_2(x,y)),\ s(p_2(x,y))),\ p_1(x,y)))$

(g) $f(x) = \lfloor \log_2(x+1) \rfloor = (\mu y)[2^{y+1} > x+1]$
$= (\mu y)[g(x,y) = 0]$ where

$g(x,y) = 1 \dot{-} gr(2^{y+1}, x+1)$
$= monus(c_1(p_1(x,y)),\ gr(exp(c_2(p_1(x,y)),\ s(p_2(x,y))),\ s(p_1(x,y))))$

9 (a)

$$\begin{aligned}
A(1,y) &= A(1, (y-1)+1) \\
&= A(1-1, A(1, y-1)) \\
&= A(0, A(1, y-1)) \\
&= A(1, y-1) + 1 \\
&= A(1, (y-2)+1) + 1 \\
&= A(0, A(1, y-2)) + 1 \\
&= A(1, y-2) + 1 + 1 \\
&\vdots \\
&= A(1, y-y) + \overbrace{1 + 1 + \cdots + 1}^{y \text{ times}} \\
&= A(1,0) + y \\
&= A(0,1) + y \\
&= (1+1) + y \\
&= y + 2 .
\end{aligned}$$

6. $f(0) = 2^0 = 1$

$f(n+1) = 2^{n+1} = 2^n + 2^n = Add(f(n), f(n))$

From this we can see that $f(n)$ is prim. recursive

$f =$ prim. rec. $(s \circ 0, \ ADD \circ [I_2^{(2)}, I_2^{(2)}])$

7. $g(x,0) = x^0 = 1$

$g(x, y+1) = x^{y+1} = x^y \cdot x$

From this we can see that $g(x,y) = x^y$ is prim. recursive

$g =$ prim. rec. $(s \circ z, \ MULT \circ [I_3^{(3)}, I_1^{(3)}])$

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9 (a) $A(1,y) = y+2$ (done on page 40)

(b)
$$
\begin{aligned}
A(2,y) &= A(2,(y-1)+1) = A(2-1, A(2,y-1)) \\
&= A(1, A(2,y-1)) = A(2,y-1)+2 \quad \text{by (a)} \\
&= A(2,(y-2)+1)+2 = A(2-1, A(2,y-2))+2 \\
&= A(1, A(2,y-2))+2 = (A(2,y-2)+2)+2 \quad \text{by (a)} \\
&= \ldots \\
&\vdots \\
&= A(2, y-y) + \overbrace{2+2+\cdots+2}^{y \text{ times}} \\
&= A(2,0) + 2y = A(1,1)+2y \\
&= 1+2+2y \quad \text{by (a)} \\
&= 2y+3
\end{aligned}
$$

(c)
$$
\begin{aligned}
A(3,y) &= A(3,(y-1)+1) = A(3-1, A(3,y-1)) \\
&= A(2, A(3,y-1)) = 2A(3,y-1)+3 \quad \text{by (b)} \\
&= 2[A(3,y-1)+3]-3 \\
&= 2[A(3,(y-2)+1)+3]-3 \\
&= 2[A(2, A(3,y-2))+3]-3 \\
&= 2.[2A(3,y-2)+3+3]-3 \quad \text{by (b)} \\
&= 2.2.[A(3,y-2)+3]-3 \\
&\vdots \\
&= \overbrace{2.2.2\ldots2}^{y \text{ times}}.[A(3,y-y)+3]-3 \\
&= 2^y.[A(3,0)+3]-3 \\
&= 2^y.[A(2,1)+3]-3 \\
&= 2^y.[(2.1+3)+3]-3 \\
&= (2^y.8)-3 = 2^{y+3}-3.
\end{aligned}
$$

10.(a) $A(4,1) = A(4, 0+1) = A(3, (A\ 4,0))$

$= 2^{A(4,0)+3} - 3$ by 9(c)

$= 2^{A(3,1)+3} - 3$

$= 2^{2^{1+3} - 3+3} - 3$ by 9(c)

$= 2^{16} - 3$

(b) $A(4,2) = A(4, 1+1) = A(3, A(4,1))$

$= 2^{A(4,1)+3} - 3$

$= 2^{2^{16} - 3+3} - 3$ by 10(a)

$= 2^{2^{16}} - 3$

11. In general

$$A(4,y) = \underbrace{2^{2^{\cdot^{\cdot^{2^{16}}}}}}_{y \text{ times}} - 3 = \underbrace{2^{2^{\cdot^{\cdot^{2^{2}}}}}}_{y+3 \text{ times}} - 3.$$

15 (a) $(\mu y)[g(x,y)=0]$

$=$ smallest y such that $x \cdot y = 0$

$= 0$ because $x \cdot 0 = 0$ & 0 is the smallest value of y with $x \cdot y = 0$.

domain $= \mathbb{N}$.

(b) $(\mu y)\,[2^x + y - 3 = 0]$

$=$ smallest y such that $2^x + y - 3 = 0$

$$= \begin{cases} 3-2^0 & \text{if } x=0 \quad \text{bec. } 2^0+0-3 \neq 0 \;\&\; 2^0+1-3 \neq 0 \\ 3-2^1 & \text{if } x=1 \quad \text{bec. } 2^1+0-3 \neq 0 \\ \text{undefined} & \text{if } x>1 \end{cases}$$

$$= \begin{cases} 2 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ \text{und} & \text{if } x>1 \end{cases}$$

domain $= \{0,1\}$. (The textbook is sloppy - it should not use "$-$". It should use "$\dot{-}$" (monus) but it didn't.)

(c) $(\mu y)\left(\left\lfloor \frac{x-1}{y+1} \right\rfloor = 0\right)$

$$= \begin{cases} \text{und.} & \text{if } x=0 \quad \text{because } \left\lfloor \frac{-1}{y+1} \right\rfloor = -1 \text{ for all } y \geq 0 \\ 0 & \text{if } x=1 \quad \text{because } \left\lfloor \frac{1-1}{0+1} \right\rfloor = 0 \\ x-1 & \text{if } x \geq 2 \quad \text{because } \left\lfloor \frac{x-1}{y+1} \right\rfloor \neq 0 \text{ for all } x \geq 2 \text{ and } y \leq x-2 \end{cases}$$

and $\left\lfloor \frac{x-1}{x-1+1} \right\rfloor = 0$ for all $x \geq 2$

$$= \begin{cases} \text{und} & \text{if } x=0 \\ x-1 & \text{if } x>0. \end{cases}$$

domain $= \mathbb{N} - \{0\}$

(d) $(\mu y)\left(x\ [\mathrm{mod}(y+1)] = 0\right) = \begin{cases} 0 & \text{if } x=0 \text{ bec. } 0\,(\mathrm{mod}\,1)=0 \\ 0 & \text{if } x>0 \text{ bec. } x\,(\mathrm{mod}\,1)=0 \end{cases}$

$= 0$ if $x \in \mathbb{N}$

domain $= \mathbb{N}$. (A rather silly question)

END