#5. (a) Use induction. Let $P(n)$ be the proposition: "If $L_1, \ldots, L_n$ are regular, then so is $\bigcup_{i=1}^{n} L_i$".
Then $P(2)$ is true by the Closure theorem.
Now suppose $P(k)$ is true.
Let $L_1, \ldots, L_k, L_{k+1}$ be any $k+1$ regular languages. Then

$$\bigcup_{i=1}^{k+1} L_i = \underbrace{\left(\bigcup_{i=1}^{k} L_i\right)}_{\substack{\text{reg.} \\ \text{by ind. hyp.}}} \cup \underbrace{L_{k+1}}_{\text{reg}} \quad \text{is regular.}$$

So $P(k) \Rightarrow P(k+1)$. Hence $P(k)$ is true for all $k \geq 2$. So the results follow.

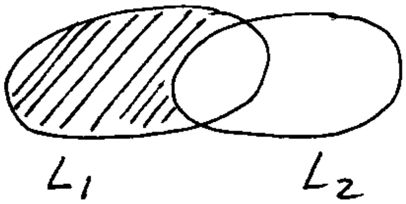
(b) The situation for intersection is entirely similar.

#6. $S_1 \ominus S_2 = (S_1 - S_2) \cup (S_2 - S_1)$
Now if S_1 & S_2 are regular, then $S_1 \ominus S_2$ will also be regular because $(S_1 - S_2)$ & $(S_2 - S_1)$ will be regular by the Closure theorem and so $(S_1 - S_2) \cup (S_2 - S_1)$ will then be regular by the Closure theorem. Hence reg. lang. are closed under symmetric differences.

#7 <u>Hint</u>: $nor(L_1, L_2) = \bar{L}_1 \cap \bar{L}_2$.
Now use the Closure theorem.

#12 Yes. First observe that $L_2 = (L_1 \cup L_2) - (L_1 - L_2)$
Now $L_1 - L_2$ is finite, so $L_1 - L_2$ is regular. And $L_1 \cup L_2$ is given as regular. $\therefore$ $(L_1 \cup L_2) - (L_1 - L_2)$ is reg. by Clos. Thm.

#12 Hence L_2 is regular.



$L_2 = (L_1 \cup L_2) - (L_1 - L_2)$

$L_1 - L_2 \subseteq L_1$. So

$L_1 - L_2$ is finite.

#13 Let Σ be the alphabet on which L is based. Then

$$L_1 = \{uv : u \in L \ \& \ |v| = 2\}$$
$$= L.\Sigma.\Sigma$$

Now any alphabet is finite, so Σ is regular. Hence $L.\Sigma.\Sigma$ is regular by the Closure Theorem. So L_1 is regular

#14 $\{uv : u \in L, v \in L^R\} = L.(L^R)$

Now use the closure theorem

#16. No. Let $L_1 = \{a^n : n \geq 0\}$ and $L_2 = \{a^p : p \text{ is prime}\}$.

Then $L_1.L_2 = \{a^n : n \geq 0\}.\{a^p : p \text{ is prime}\}$

$$= \{a^{n+2} : n \geq 0\} = \underline{a}\,\underline{a}\,\underline{a}^*$$

So L_1 & $L_1.L_2$ are both regular. It will be show later that L_2 is non-regular by using various means. (See supplementary problems also)

#26. See class notes.

#1 $L_1 \subseteq L_2 \Leftrightarrow L_1 - L_2 = \emptyset$. Since L_1 & L_2 are regular we can find dfa's M_1 & M_2 for L_1 & L_2. Using M_1 & M_2 we can algorithmically get a dfa M for $L_1 - L_2 = (\bar{L}_1 \cup L_2)^c$. Now $L_1 - L_2 = \emptyset \Leftrightarrow L(M) = \emptyset \Leftrightarrow$ there is no path from the initial state of M to an accepting state of M (and this can be checked algorithmically)

#2 Since L is regular, we can find a dfa M such that $L(M) = L$. Now $\lambda \in L \Leftrightarrow q_0 \in F(M)$ (i.e., if the initial state of M is an accepting state also). Since this can be algorithmically checked there is an algorithm to tell if $\lambda \in L$.

#5 Since L is reg., we can find a dfa M such that $L(M) = L$. Let M^R be the machine obtained by making all accepting states in M into initial states of M^R & all initial states in M into accepting states of M^R. Then it can be algorithmically checked by Thm 4.7 if $L(M) = L(M^R)$. L is palindromic $\Leftrightarrow L = L^R \Leftrightarrow L(M) = L(M^R)$.
Oh! You also have to reverse the arrows in M to get M^R.

#6. Make the machine M^R as above. Then L has a string w such that $w^R \in L \Leftrightarrow w \in L(M)$ & $w^R \in L(M)$ $\Leftrightarrow w \in L(M)$ & $w \in L(M^R) \Leftrightarrow w \in L(M) \cap L(M^R)$.
So check algorithmically if $L(M) \cap L(M^R) \neq \emptyset$.

#1 Hint: The idea is to look for the last occurrence of a repetition, in an accepting sequence. Following the proof on p. 119, look at the sequence

$$q_0, q_i, q_j, \ldots, q_f.$$

At least one state must be repeated, and such a repetition must start no later than the n-th move from the end. From this the proof will follow.

#3(a) Let $L = \{\omega \in \{a,b\}^* : n_a(\omega) = n_b(\omega)\}$. Then L is infinite. Supp. L is regular. Let m be as in the Pumping Lemma. Choose $u = a^m b^m$. Then $u = xyz$ as in the Lemma.

$|xy| \leq m \Rightarrow y$ consists only of a's

$n_a(xyyz) > m$ but $n_b(xyyz) = m$

$\therefore$ $xyyz \notin L$. But $xy^n z \in L$ for all $n \geq 0$ by the lemma. Hence we have a contradiction. So L cannot be regular.

(b) $L^* = L$. So L^* is also non-regular.

#4 (a) Use the Pumping Lemma with $u = a^m b\, a^{m+1}$

(b) If L is regular, so is $\bar{L}$. And if $\bar{L}$ is regular so is $\bar{L} \cap \underline{a}^* \underline{b}^* \underline{a}^* = \{a^n b^\ell a^k : k = n+\ell\} = L_a$

But L_a is not reg. by (a). So L is not regular.

SECTION 4.3 p. 123

\#4 (c) Use the Pumping Lemma with $\mu = a^m b^m a^m$

(d) Use the Pumping Lemma with $\mu = a^m b^m$

(e) If L is regular, then $\bar{L}$ will be regular. But $\bar{L}$ is non-regular from Prob. #3. So L is not regular.

(f) Using the Pumping Lemma with $\mu = a^m b a^m b$ will give you the result.

5.(a) $L = \{a^p : p \text{ is prime}\}$ $\qquad$ primes $= \{2,3,5,7,11,\dots\}$

Apply the Pumping Lemma with $\mu = a^p$ where p is a prime $\geq m$. Then

$\mu = xyz$ as in the PL

So $$\mu = a^{|x|} \cdot a^{|y|} \cdot a^{|z|} = a^{|x|+|z|} \cdot a^{|y|}$$

Now by the P.L. $\quad xy^n z = a^{|x|+|z|} \cdot a^{n|y|} \in L$

So

$(|x|+|z|) + k|y|$ will always be prime.

Let $\alpha = |x|+|z|$ & $\beta = |y|$. Then $\alpha + k\beta$ will always be prime. But

if $\alpha = 0$ or 1, we get a contradiction with $k=0$

And if $\alpha \geq 2$, we get a contradiction with $k=\alpha$, because $\alpha + \alpha\beta = \underbrace{\alpha}_{\geq 2}\underbrace{(1+\beta)}_{\geq 2}$ which is not prime

$\therefore$ L is not regular.

\#5 (b) $\bar{L}$ is not regular by 5(a). So L is not regular also

(c) Apply P.L. with $\mu = a^{(m^2)}$

(d) Apply P.L with $\mu = 2^m$

\#14 False. Let $L_1 = \{a^n b^n : n \geq 0\}$ and $L_2 = \{a,b\}^* - L_1$. Then L_1 and L_2 are both non-regular but $L_1 \cup L_2 = (\underline{a} \cup \underline{b})^*$ is regular.

\#17 Yes. $L = L_1 \cap (L_2^R)$. Now use Closure Theorem.

\#21 Let $L_n = \{a^n b^n\}$. Then L_n is reg. for each $n \geq 0$. But $L = \bigcup_{n=0}^{\infty} L_n = \{a^i b^i : i \geq 0\}$ is not regular.

\#23 No. Let $L = \{a,b\}^* - \{a^n b^n : n \geq 0\}$. Then L is a non-regular language. Let

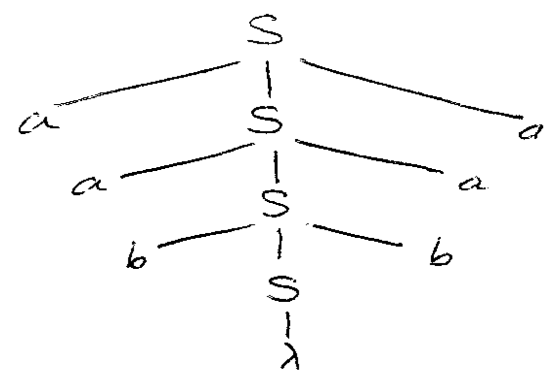
$$L_i = \{a,b\}^* - \{a^i b^i\} \quad \text{for } i \geq 0.$$

Then each L_i is regular but $\bigcap_{n=0}^{\infty} L_i = L$ which is non-regular.

\#24 No. Let $L_1 = \{a,b\}^*$ & $L_2 = \{a^n b^n : n \geq 1\}$. Then L_1 is reg. & $L_1 \cup L_2 = \{a,b\}^*$ is also regular. But L_2 is not regular. (See #11 Sec. 4.1)

SECTION 5.1 p. 133

#2 The derivation tree for aabbaa is given on the right



#7 (a) $S \to ASb \mid AAA, \quad A \to a \mid \lambda$

(b) $S \to A \mid B, \quad A \to aAb \mid aA \mid \lambda, \quad B \to aBb \mid Bb \mid bb$

$\{a^n b^m : n \geq m\}$ $\quad \{a^n b^m : n \leq m-2\}$

(c) $S \to A \mid Bb, \quad A \to aaAb \mid aA \mid a$ $\quad \{a^n b^m : n \geq 2m+1\}$

$B \to DDBb \mid D, \quad D \to a \mid \lambda$ $\quad \{a^n b^m : n \leq 2(m-1)+1\}$

d) $S \to aSbbB \mid \lambda, \quad B \to b \mid \lambda$.

#8 (a) $S \to A \mid B, \quad A \to Ac \mid D, \quad D \to aDb \mid \lambda$ $\quad n = m$

$B \to aB \mid E, \quad E \to GEc \mid \lambda, \quad G \to b \mid \lambda$ $\quad m \leq k$

(b) $S \to A \mid B, \quad A \to Ac \mid D, \quad D \to aDb \mid \lambda$ $\quad n = m$

$B \to aB \mid E, \quad E \to bEc \mid F \mid G, \quad F \to bF \mid b$ $\quad m > k$

$G \to Gc \mid c$ $\quad m < k$

(c) $S \to aSc \mid T, \quad T \to bTc \mid \lambda$

(d) $S \to aSc \mid T, \quad T \to bTcc \mid \lambda$.

#13 (a) L^2 is generated by $S \to AA$, $A \to aAb \mid \lambda$

(b) L^k is generated by $S \to \underbrace{AA\cdots A}_{k \text{ times}}$, $A \to aAb \mid \lambda$

(c) L^* is generated by $S \to AS \mid \lambda$, $A \to aAb \mid \lambda$

#20 Any derivation of aa.bbabb.a would have to start with

$$S \Rightarrow aaB \Rightarrow aaAa \Rightarrow aabBba$$
$$\Rightarrow aabAaba$$

and this can never lead to aab.bab.ba

#22 $S \to [S] \mid (S) \mid SS \mid \lambda$

#23 $S \to \underline{\lambda} \mid \emptyset \mid \underline{a} \mid \underline{b} \mid (S+S) \mid (S.S) \mid (S^*)$

#24 Let $V = \{X, Y\}$ and $T = \{A, B, C, a, b, \dot{\to}\}$

The productions are given below:

$$S \longrightarrow X \dot{\to} Y, \quad X \to A \mid B \mid C, \quad Y \to A \mid B \mid C \mid a \mid b \mid YY$$

"$\dot{\to}$" denotes arrows from CFG's

$\to$ denotes arrows from our grammar.

#25 Hint: Just keep applying the appropriate productions to get the rightmost needed letter to get the right most derivation. Same thing for the left most derivation except you look for left most needed letter.

SECTION 5.2 p. 145.

#1 $S \to aA / b$, $A \to aB$, $B \to aB / b$

#2 $S \to aA$, $A \to aAB / b$ $B \to b$

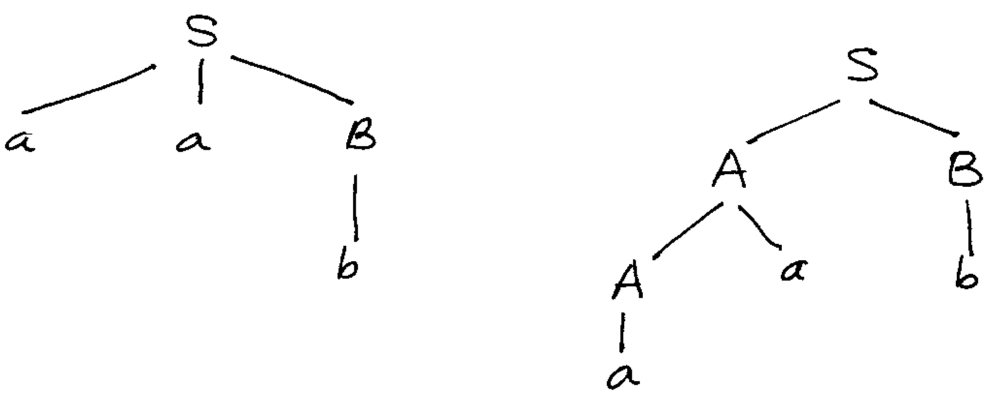
#4 Hint: If G is an s-grammar, then each string φ in $L(G)$ will have a unique leftmost derivation.

#5 $\underbrace{A}_{|V|\text{ choises}} \to \underbrace{a}_{|T|\text{ choices}} x$, $x \in V^*$

Maximum size of $|P| = |V| \cdot |T|$

#6 The string aab has two left-most derivations. So the grammar is ambiguous.

1. $S \Rightarrow aaB \Rightarrow aab$
2. $S \Rightarrow AB \Rightarrow AaB \Rightarrow aaB \Rightarrow aab$



These are the two different derivation trees.

9. If L is a regular language then we can find a DFA which accepts L. Now if we convert this DFA into a RLG there will never be a choice of productions because the DFA was deterministic. So we will get an unambiguous RLG for L. This means L is not inherently ambiguous.

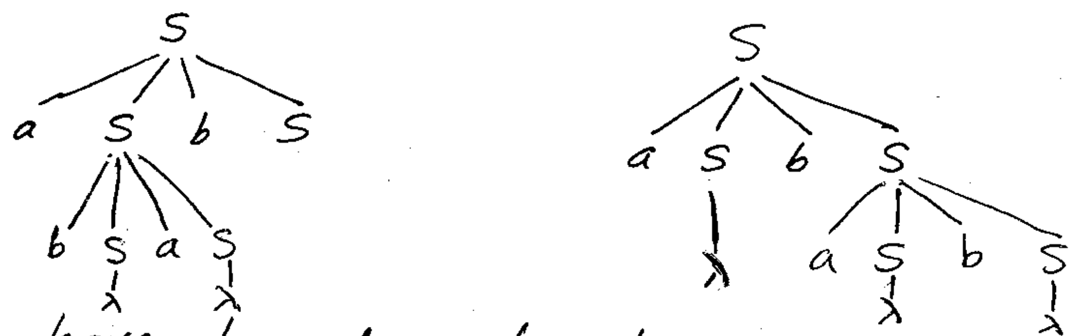
11. Yes. Let G be the grammar with productions

$$S \to aA, \quad S \to ab, \quad A \to b$$

Then ab has two leftmost derivations in G. So G is ambiguous.

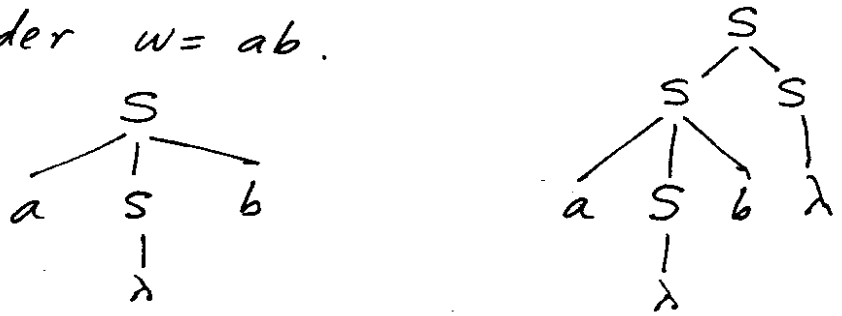
G: $S \to aSbS \mid bSaS \mid \lambda$

13(a) Consider the string $w = abab$



We have two derivation trees. So grammar is ambiguous.

(b) Consider $w = ab$.



G: $S \to aSb \mid SS \mid \lambda$ $\quad\therefore$ G is ambiguous