MAC 2313 - Multivariable Calc. Florida Int'l Univ.

TEST #3 - Spring 2004 TIME: 100 min.

Answer all 8 questions. An unjustified answer will receive little or no credit. NO CALCULATORS OR FORMULA SHEETS ARE ALLOWED. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(14) 1. (a) Let $\underset{\sim}{F} = \langle x^2 z, xy^2, yz^2 \rangle$. Find $div(\underset{\sim}{F})$ & $curl(\underset{\sim}{F})$.

(b) Let $x = u^3$, $y = v \sin(w)$ and $z = v \cos(w)$. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

(12) 2. (a) Make a sketch of the solid region G bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 9$ that is above the xy plane.

(b) Find the volume of G by using spherical coordinates.

(12) 3. (a) Make a sketch of the portion σ of the surface $z = 1 - x^2 - y^2$ that lies above the xy plane.

(b) Find the surface **area** of this portion σ.

(14) 4. (a) Determine whether or not the vector field $\underset{\sim}{F} = \langle 2xz + y^2, z + 2xy, x^2 + y \rangle$ is conservative.

(b) Let $\underset{\sim}{F} = \langle 8x + y^2 \cos x, 2y \sin x - 3y^2 \rangle$. Find all the scalar potentials $\varphi(x,y)$ such that $\underset{\sim}{F} = \underset{\sim}{\nabla} \varphi(x,y)$.

PLEASE TURN OVER

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(12) 5. Let C be the parametric curve given by

$$\underset{\sim}{r}(t) = \langle 2t, \cos t, \sin t \rangle; \quad 0 \le t \le \pi/2 .$$

Find the work done in moving a particle from $\langle 0,1,0\rangle$ to $\langle \pi, 0, 1\rangle$ in the vector field $\underset{\sim}{F} = \langle 3x, 4y, 3yz\rangle$.

(12) 6. A thin wire in the form of a helix

$$\underset{\sim}{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad 0 \le t \le \pi$$

has density $\delta(x,y,z) = \frac{1}{3}x^2y$ at $\langle x,y,z\rangle$. Find the total mass of the wire by using a line integral.

(12) 7. Let σ be the parametric surface given by

$$\underset{\sim}{r}(u,v) = \langle u\cos(v), u\sin(v), 2u\rangle \quad 0 \le u \le 1, \ 0 \le v \le \pi/2.$$

If the density of the surface at the point $\langle x,y,z\rangle$ is given by $\delta(x,y,z) = 3(x+z)$, find the total mass of σ by using a surface integral.

(12) 8. Let σ be the parametric surface given by

$$\underset{\sim}{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle; \quad 0 \le u \le 1, \ 0 \le v \le 2\pi$$

with the normals oriented in the same direction as $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$. Also let $\underset{\sim}{F}(\underset{\sim}{r}) = \langle x, y, 6z\rangle$.

Find the total flux of the vector field $\underset{\sim}{F}$ across σ.

Hint: Remember $\underset{\sim}{r} = \underset{\sim}{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ in #7 & #8. So you know x, y, z in terms of u & v.

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Solutions to Test # 3 Spring 2004

1(a) $\operatorname{div}(\underset{\sim}{F}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle x^2 z, xy^2, yz^2\rangle$

$$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(y^2) = 2(xz + xy + yz)$$

$$\operatorname{curl}(\underset{\sim}{F}) = \underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & yz^2 \end{vmatrix}$$

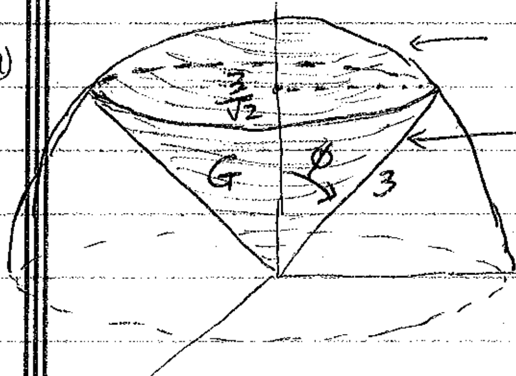
$$= \left\langle \frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial z}(xy^2),\ \frac{\partial}{\partial z}(x^2 z) - \frac{\partial}{\partial x}(yz^2),\ \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x^2 z)\right\rangle$$

$$= \langle z^2 - 0,\ x^2 - 0,\ y^2 - 0\rangle = \langle z^2, x^2, y^2\rangle$$

(b) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & \sin(w) & v\cos(w) \\ 0 & \cos(w) & -v\sin w \end{vmatrix}$$

$$= 3u^2\left(-v\sin^2 w - v\cos^2 w\right) = -3u^2 v.$$

2(a)



$z^2 + x^2 + y^2 = 9$

$z = \sqrt{x^2+y^2}$

$\rho = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2 + y^2$

$\therefore\ z^2 + r^2 = 9\ \&\ z = \sqrt{r^2}$

$\therefore\ 2z^2 = 9 \Rightarrow z = 3/\sqrt{2}$

and $r = 3/\sqrt{2}$

$\therefore\ \phi$ varies from 0 to $\frac{\pi}{4}$.

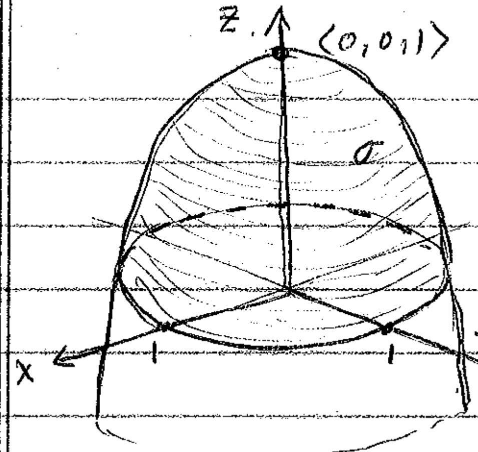
ρ varies from 0 to 3

(b) $$V = \iiint_G 1\, dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{3} 1.\ \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^3 .\ \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 9\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} 9\left[-\cos\phi\right]_0^{\pi/4} d\theta = \left[\int_0^{2\pi} d\theta\right]\cdot\left[9\left(1 - \frac{\sqrt{2}}{2}\right)\right] = 9\pi\left(2 - \sqrt{2}\right).$$

3(a)



$$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\; dA$$
$$= \sqrt{1+(-2x)^2+(-2y)^2}$$
$$= \sqrt{1+4(x^2+y^2)} = \sqrt{1+4r^2}$$

$z = 1-x^2-y^2$ $\qquad R := x^2+y^2 \le 1$

(b) $$S = \iint_\sigma 1\, dS = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\; dA$$
$$= \iint_R \sqrt{1+4r^2}\cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 \frac{1}{8}(4r^2+1)^{1/2}\cdot 8r\,dr\,d\theta$$
$$= \left[\int_0^{2\pi} d\theta\right]_0^{2\pi}\left[\frac{1}{8}(4r^2+1)^{3/2}\cdot\frac{2}{3}\right]_0^1$$
$$= \frac{2\pi\cdot 2}{8\cdot 3}\left(5^{3/2}-1\right) = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$$

$$\boxed{\begin{array}{l} \text{Put } u = 4r^2+1 \\ du = 8r\,dr \\ \int (4r^2+1)^{1/2}\cdot r\,dr \\ = \int u^{1/2}\cdot\frac{1}{8}\,du \\ = \frac{1}{8}\frac{2}{3}u^{3/2}+C \end{array}}$$

4(a) $$\operatorname{curl}(\underset{\sim}{F}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz+y^2 & z+2xy & x^2+y \end{vmatrix} = \langle 1-1,\ 2x-2x,\ 2y-2y\rangle$$
$$= \langle 0,0,0\rangle = \underset{\sim}{0}$$

$\therefore$ $\underset{\sim}{F}$ is a conservative vector field.

(b) Let $\underset{\sim}{F} = \underset{\sim}{\nabla}\phi$. Then $\langle 8x+y^2\cos x,\ 2y\sin x - 3y^2\rangle = \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right\rangle$

$\therefore$ $\frac{\partial\phi}{\partial x} = 8x + y^2\cos x \Rightarrow \phi = 4x^2 + y^2\sin x + g(y)$

and $\frac{\partial\phi}{\partial y} = 2y\sin x - 3y^2 \Rightarrow \phi = y^2\sin x - y^3 + f(x)$

$\therefore$ $4x^2+y^2\sin x + g(y) = y^2\sin x - y^3 + f(x)$

$\therefore$ $f(x) = 4x^2 + C$ and $g(y) = -y^3 + C$

where C is an arbitrary constant.

Hence $\phi(x,y) = 4x^2 + y^2\sin x + g(y)$
$= 4x^2 + y^2\sin x - y^3 + C.$

5. $\frac{d(r)}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = \langle 2, -\sin t, \cos t \rangle$

$\underset{\sim}{F}(\underset{\sim}{r}) = \langle 3x, 4y, 3yz \rangle = \langle 6t, 4\cos t, 3\cos t \sin t \rangle$

$W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{\pi/2} \left(\underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \frac{dr}{dt} \right) dt$

$= \int_0^{\pi/2} (12t - 4\sin t \cos t + 3 \sin t \cos^2 t) \, dt$

$= \left[6t^2 - \cancel{4}^2 \cdot \frac{1}{\cancel{2}} \sin^2 t - \cancel{3} \cdot \frac{1}{\cancel{3}} \cos^3 t \right]_0^{\pi/2}$

$= \left(6 \cdot \frac{\pi^2}{4} - 2 \cdot 1 - 0 \right) - (0 - 0 - 1) = \frac{3\pi^2}{2} - 1.$

6. $d\underset{\sim}{r}/dt = \langle -3\sin t, 3\cos t, 4 \rangle$

$\| d\underset{\sim}{r}/dt \| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (4)^2} = \sqrt{9+16} = 5.$

$M = \int_C \delta(\underset{\sim}{r}) \, dl = \int_{t=0}^{\pi} 3x^2 y \left\| \frac{d\underset{\sim}{r}}{dt} \right\| dt$

$= \int_0^{\pi} \frac{1}{3} (3\cos t)^2 (3\sin t) \cdot 5 \cdot dt = 15 \int_0^{\pi} \cos^2 t \sin t \, dt$

$= 15 \left[-\frac{1}{3} \cos^3 t \right]_0^{\pi} = 15 \left[-(-1) - (-1) \right] = 30.$

7. $\frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \cos v & \sin v & 2 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -2u\cos v, -2u\sin v, u \rangle$

$\left\| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right\| = \sqrt{(-2u\cos v)^2 + (-2u\sin v)^2 + u^2} = u\sqrt{5}$

$M = \iint_\sigma \delta(\underset{\sim}{r}) \, dS = \iint_R \delta(\underset{\sim}{r}) \left\| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right\| dA_{uv}$

$= \int_{u=0}^{1} \int_{v=0}^{\pi/2} (3u\cos v + 2u) \cdot u\sqrt{5} \cdot dv \, du$

$= \int_0^1 \int_0^{\pi/2} u^2 (\cos v + 2) \cdot \sqrt{5} \, dv \, du$

$= \sqrt{5} \int_0^1 u^2 du \cdot \int_0^{\pi/2} \cos(v) + 2 \, dv = \left[\frac{u^3}{3} \right]_0^1 \cdot \left[\sin(v) + 2v \right]_0^{\pi/2}$

$= \sqrt{5} \cdot (\pi + 1)$

8. $\underset{\sim}{r}(u,v) = \langle u\cos v,\ u\sin v,\ u^2\rangle$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \langle -2u^2\cos v,\ -2u^2\sin v,\ u(\cos^2 v + \sin^2 v)\rangle$$

$$= \langle -2u^2\cos v,\ -2u^2\sin v,\ u\rangle$$

$$\underset{\sim}{F}(\underset{\sim}{r}) = \langle x, y, 6z\rangle = \langle u\cos v,\ u\sin v,\ 6u^2\rangle$$

$$\text{Flux} = \iint_\sigma \underset{\sim}{F}\cdot d\underset{\sim}{S} = \iint_R \left(-2u^3\cos^2 v - 2u^3\sin^2 v + 6u^3\right) dA_{uv}$$

$$= \int_{u=0}^{1}\int_{v=0}^{2\pi} 4u^3\, dv\, du = \int_0^1 4u^3 du \cdot \int_0^{2\pi} dv$$

$$= \left[u^3\right]_0^1 \cdot \left[v\right]_0^{2\pi} = 2\pi.$$