PROJECT 0.4

There are six functions including f and g:

$f \circ g(x) = \frac{x}{x - 1}$, $g \circ f(x) = \frac{2x - 3}{x - 2}$, $g \circ g(x) = x$,
$f \circ f(x) = \frac{2x - 3}{x - 1}$

The easiest way to show that this list is complete is to show that any composition of three functions reduces to one of the functions on this list.

PROJECT 1.3

Case 1: $a \geq 2$. Then $a_2 = \sqrt{a + a_1} = \sqrt{a + a} = \sqrt{2a} \leq a = a_1$.
The sequence is decreasing as is shown by the induction step:
$a_{n+1} = \sqrt{a + a_n} \leq \sqrt{a + a_{n-1}} = a_n$. The sequence is bounded from below by 0.

Case 2: $0 < a < 2$. Then $a_2 = \sqrt{a + a_1} = \sqrt{a + a} = \sqrt{2a} > a = a_1$.
The sequence is increasing as shown by the induction step:

$a_{n+1} = \sqrt{a + a_n} \leq \sqrt{a + a_{n-1}} = a_n$. Also the sequence is bounded from above by 2. To see that, $a_2 = \sqrt{a + a_1} = \sqrt{a + a} = \sqrt{2a} \leq 2$ and $a_{n+1} = \sqrt{a + a_n} \leq \sqrt{a + 2} \leq 2$.

In both case 1 and case 2, the limit L of the sequence must satisfy the equation $L = \sqrt{L + a}$ or $L = \frac{1 + \sqrt{1 + 4a}}{2}$