32. Let \( Y \) be the set of ordinals less than the first uncountable ordinal; i.e., \( Y = \{ x \in X : x < \Omega \} \). Show that every countable subset \( E \) of \( Y \) has an upper bound in \( Y \) and hence a least upper bound.

Proof: Let’s denote the order on \( Y \) inherited via Proposition 8 of Chapter 1 by \(<\). Suppose that \( E \) is a countable, non-empty subset of \( Y \). For each \( y \in E \), let

\[
S_y = \{ x \in Y : x \leq y \} = \{ x \in Y : x < y \} \cup \{ y \}.
\]

From Proposition 8 of Chapter 1, \( S_y \) must be countable for each \( y \in E \). Thus, \( S = \bigcup \{ S_y : y \in E \} \), a countable union of countable sets, must be countable. From the proof of Proposition 8 of Chapter 1, \( Y \) must be uncountable. Consequently, \( Y \sim S \) must be non-empty. Observe that from the simplicity of the well ordering and our friendly De Morgan’s laws, \( Y \sim S = \cap \{ T_y : y \in E \} \), where \( T_y = \{ x \in Y : x > y \} \) for each \( y \in E \). Thus, if \( y_0 \in Y \sim S \) and \( y \in E \), then \( y < y_0 \). Thus, any element of \( Y \sim S \) will be an upper bound for \( E \). Evidently, from the well ordering, the set of upper bounds must have a least element, and we are finished.