A **differential equation** is an equation with a derivative in it. For example, $y' - y = x$ is a differential equation. Solving a differential equation means to find all the functions that make the equation a true sentence. Generally, a differential equation has infinitely many solutions but, when we add an **initial condition** like y(0) =1 (when x = 0, y = 1), only one of the infinitely many solutions will satisfy the initial condition. A differential equation, together with an initial condition, is called an **initial value problem**. For example, $y' - y = x$, y(0) = 1 is an initial value problem. We can use power series to solve initial value problems.

Example: Use series to solve the initial value problem $y' - y = x$, y(0) = 1. Find the first 4 non-zero terms of the solution.

Solution: We assume a solution of the form $y = c_0 + c_1x + c_2x^2 + c_3x^3 + ...$

Differentiating gives us $y' = c_1 + 2c_2x \ \ + 3c_3x^2 + ...$

Substituting into the given differential equation:

$c_1 + 2c_2x \ \ + 3c_3x^2 ...$ - $(c_0 + c_1x + c_2x^2 + c_3x^3 + ...)$ = x

Combining like terms:

$(c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + ... = x$

The right side of the equation can be thought of as a power series where all the coefficients except one have coefficient zero:

$(c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + ... = 0 + 1x + 0x^2 + ... + 0x^{n-1} + ...$

Equating the coefficients:

$c_1 - c_0 = 0$

$2c_2 - c_1 = 1$

$3c_3 - c_2 = 0$

Our initial condition, when substituted into the series for y, tells us $c_0 = 1$

Solving each equation:

$c_1 = c_0 = 1$

$$c_2 = \frac{1 + c_1}{2} = \frac{1 + c_0}{2} = \frac{1+1}{2} = 1$$

$$c_3 = \frac{c_2}{3} = \frac{1}{3}$$

So our solution is the series $y = 1 + x + x^2 + \frac{1}{3}x^3 + ...$

Use series to solve the following initial value problems. Find the first 4 non-zero terms of the solution.

1. $y' + y = 0$, y(0) = 1
2. $y' - y = 1$, y(0) = 0