Section 1.2

(4) All but the third string are in $L^*$.

(5) If $L = \{a^n b^{n+1} : n \geq 0\}$ then $L \neq L^*$. For example, $\lambda \in L^*$ but $\lambda \notin L$.
If $L = \{w : n_a(w) = n_b(w)\}$ then $L = L^*$.

(6) There are no languages $L$ such that $\overline{L} = \overline{L}^*$. One way to see this is that for any language $L$, $\lambda \in \overline{L}^*$ but $\lambda \notin \overline{L}$.

(8) (a) $S \to BaB$
    $B \to bB | \lambda$

(b) $S \to AaA$
    $A \to aA | bA | \lambda$

(c) $S \to B | BaB | BaBaB | BaBaBaB$
    $B \to bB | \lambda$

(9) $L(G) = \{(ab)^n \mid n \geq 0\}$.

(10) $L(G) = \emptyset$ because there is no way to derive a string that doesn’t contain one of the auxiliary symbols $S, A, B$.

(11) (a) $S \to aSb | B$
    $B \to bB | b$

(b) $S \to aSbb | \lambda.$
    $A \to aAb | \lambda$

(c) $S \to aaaAb$

(d) $S \to aaaA$
    $A \to aAb | \lambda$

(e) $S \to S_1 S_2$
    $S_1 \to aS_1 b | B$
    $B \to bB | b$
    $S_2 \to aS_2 bb | \lambda$
    $S_2 \to aS_2 bb | \lambda$

(f) $S \to S_1 | S_2$
    $S_1 \to aS_1 b | B$
    $B \to bB | b$

(g) $S \to S_1 S_1 S_1$
    $S_1 \to aS_1 b | B$
    $B \to bB | b$

(h) $S \to SS | \lambda | S_1$
    $S_1 \to aS_1 b | B$
    $B \to bB | b$

(i) $L_1 - \overline{L} = L_1 \cap L_4$. But $L_1 \cap L_4 = \emptyset$.
A grammar for the empty language is: $S \to S$.

(12) (a) $S \to aaaS | \lambda.$

(c) $S \to aaaaaaS | aa | aaa | aaaa | aaaaa$

(13) $S \to aSa | bSb | aa | bb$
CLAIM Let $G$ be the grammar above. Let $L = \{ww^R \mid w \in \{a, b\}^+\}$. The $L(G) = L$.

Proof. First we show that $L(G) \subseteq L$. We prove by induction on the length of a derivation of $v$ that if $v$ is a sentential form of $G$ then $v$ has one of the following three forms: (i) $S$; (ii) $ww^R$; or (iii) $ww^R$, where $w \in \{a, b\}^+$.

Basis Step. If the derivation of $v$ has length 0 then $v = S$.

Inductive Step. Suppose $S \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_n \Rightarrow v$ is a derivation of $v$ of length $n + 1$. By induction, $v_n$ has form (i) (ii) or (iii). Since all production rules have an $S$ on the left hand side, $v_n$ cannot have form (iii). If $v_n = S$ then by examining the production rules we see that $v = aSa$ or $v = bSb$ (these are of form (ii)) or $v = aa$ or $v = bb$ (these are of form (iii)). If $v_n = wSw^R$ then by examining the production rules we see that $v = waSaw^R$ or $v = wbSbw^R$ or $v = waaw^R$ or $v = wbw^R$. Since $aw^R = (wa)^R$ and $bw^R = (wb)^R$, we are done.

Next we will prove that $L \subseteq L(G)$. We prove by induction on the length of a string $w \in \{a, b\}^+$ that $ww^R$ is a sentential form of $G$. Since $G$ contains the production rule $s \rightarrow \lambda$, this is sufficient.

Basis Step. If $|w| = 1$ then $w = a$ or $w = b$. By inspection, $aSa$ and $bSb$ are sentential forms of $G$.

Inductive Step. Suppose $|w| = n + 1$ where $n \geq 1$. Then there is a string $v$ of length $n$ such that either $w = va$ or $w = vb$. By induction $vSv^R$ is a sentential form of $G$. By inspection then $vaSav^R$ and $vbSbv^R$ are sentential forms. Since $av^R = (va)^R$ and $bv^R = (vb)^R$, we are done.

(14) (a) $S \rightarrow S_1 aS_1$  (b) $S \rightarrow aS_1 | aS | S_1 S$

$S_1 \rightarrow S_1 S_1 | \lambda | aS_1 b | bS_1 a$  

$S_1 \rightarrow S_1 S_1 | \lambda | aS_1 b | bS_1 a$

(17) No, the 2 grammars are not equivalent. The first generates $\lambda$ and the second does not.

(19) $L(G) = \{a^{2^n} : n \geq 1\}$. 