
Section 4.3

(3) Let \( L = \{ w \in \{a,b\}^* : n_a(w) = n_b(w) \} \). Then \( L \) is not regular. (Since \( L^* = L, L^* \) is also not regular either.)

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^mb^m \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^k \) for some \( k \leq k \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+k}b^m \). So \( w_i \notin L \) because \( m + k \neq m \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(4a) Let \( L = \{ a^n b^k : k \geq n + l \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^mb^m a^m \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+t}b^m \). So \( w_i \notin L \) because \( 2m \neq (m+t) + m \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(4b) Let \( L = \{ a^n b^k : k \neq n + l \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Then \( \bar{L} \cap L(a^* b^* a^*) \) is also regular since the family of regular languages is closed under compliment and intersection. Let us write \( L_1 \) for \( \bar{L} \cap L(a^* b^* a^*) \). Notice that \( L_1 = \{ a^n b^k : k = n + l \} \). We will apply the Pumping Lemma to \( L_1 \). Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^m b^m a^m \). Notice that \( w \in L_1 \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+t}b^m \). So \( w_i \notin L_1 \) because \( 2m \neq (m+t) + m \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(4c) Let \( L = \{ a^n b^k : n = l \text{ or } l \neq k \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^m b^m a^m \). Notice that \( w \in L \) (since \( n=m=l \)) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+t}b^m a^m \). So \( w_i \notin L \) because \( n = m+t \neq m = l \text{ and } l = m = k \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(4d) Let \( L = \{ a^n b^l : n \leq l \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^m b^m \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+t}b^m \). So \( w_i \notin L \) because \( m + t \neq m \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)
(4e) Let \( L = \{ w \in \{a,b\}^* : n_a(w) = n_b(w) \} \). Then \( L \) is not regular.

Proof. If \( L \) were regular then \( \overline{L} \) would be regular. But we proved in exercise (3) above that \( \overline{L} \) is not regular. \( \square \)

(4f) Let \( L = \{ ww : w \in \{a,b\}^* \} \). Then \( L \) is not regular.

Proof. Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^m ba^m b \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^k \) for some \( k \) with \( 1 \leq k \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2 z = a^{m+k} ba^m b \). So \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(5a) We did this one in class.

(5b) This follows from 5a since the family of regular languages is closed under compliments.

(5c) Let \( L = \{ a^n : n = k^2 \text{ for some } k \geq 0 \} \). Then \( L \) is not regular.

Proof. Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^{m^2} \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2 z = a^{m^2+t} \). Now \( m^2 + t \leq m^2 + m < m^2 + 2m + 1 = (m+1)^2 \). So \( m^2 + t \neq k^2 \) for any \( k \). So \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(5d) Let \( L = \{ a^n : n = 2^k \text{ for some } k \geq 0 \} \). Then \( L \) is not regular.

Proof. Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^{2^m} \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^t \) for some \( t \) with \( 1 \leq t \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2 z = a^{2^m+t} \). Now \( 2^m + t \leq 2^m + m < 2^m + 2m = 2(2^m) = 2^{m+1} \). So \( 2^m + t \neq 2^k \) for any \( k \). (In the above calculation we use the fact that, since \( m \geq 1, m < 2^m \). This can be proved by induction on \( m \).) So \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(8) Consider the statement: “If \( L_1 \) and \( L_2 \) are nonregular languages, then \( L_1 \cup L_2 \) is nonregular.” This statement is FALSE. For example let \( L_1 \) be the \( L \) from exercise (5d) above. So \( L_1 \) is nonregular. Let \( L_2 = \{a\}^* - L_1 \). Since the family of regular languages is closed under compliment, \( L_2 \) is also nonregular. But \( L_1 \cup L_2 = \{a\}^* \) which, of course, is regular.

(9a) Let \( L = \{ a^nb^ka^k : n + l + k > 5 \} \). Then \( L \) is regular. Here is a regular expression for \( L \):

\[
\begin{align*}
\text{aaaaaa}^*b^*a^* + \text{aaaaabb}^*a^* + \text{aaaaabaa}^* + \text{aabbba}^*a^* + \text{aabbaa}^* + \\
+ \text{aabbbb}^*a^* + \text{aabbaaa}^* + \text{aabbaaa}^* + \text{abbbbb}^*a^* + \text{abbbbaa}^* + \text{abbbaaa}^* + \text{abbbaaa}^* + \text{abbaaaa}^* + \\
+ \text{bbbbbba}^*a^* + \text{bbbbbba}^* + \text{bbbaaa}^* + \text{bbbaaa}^* + \text{baaaaa}^*
\end{align*}
\]
(9b) Let \( L = \{ a^nb^l : n > 5, l > 3, k \leq l \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^nb^{lm}a^{4m} \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that there are three cases for what \( y \) looks like. Either (i) \( y = a^t \) for some \( t \) with \( 1 \leq t \leq 6 \); or (ii) \( y = b^t \) for some \( t \) with \( 1 \leq t \leq m \); or (iii) \( y = a^tb^s \) for some \( t \) and \( s \) with \( 1 \leq t \leq 6 \) and \( 1 \leq s \leq m \). In Case (i), let \( i = 0 \). Then \( w_i = w_0 = xz = a^{6-t}b^{4m}a^{4m} \). Then \( w_i \notin L \) since \( 6 - t \) is not greater than 5. In Case (ii) let \( i = 0 \). Then \( w_i = w_0 = xz = a^{6}b^{4m-t}a^{4m} \). Then \( w_i \notin L \) since it is not the case that \( 4m \leq 4m - t \). In Case (iii) let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^tb^s a^tb^s z \). So again \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular.

(9c) Let \( L = \{ a^nb^l : n/l \text{ is an integer} \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^{m+1}b^{n+1} \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^k \) for some \( k \) with \( 1 \leq k \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+k}b^{n+1} \). Now \( m + k + 1 \leq m + m + 1 < 2m + 2 = 2(m + 1) \). So \( m + k + 1 \) is not a multiple of \( m + 1 \). So \( (m + k + 1)/(m + 1) \) is not an integer. So \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular.

(9d) Let \( L = \{ a^nb^l : n + l \text{ is a prime number} \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Let \( p \) be the least prime number greater than \( m \). Then let \( w = a^pb^0 = a^p \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^k \) for some \( k \) with \( 1 \leq k \leq m \). Now let \( i = p + 1 \). Then \( w_i = a^{p+k} \). Now \( p + pk = p(k + 1) \) is not a prime number. So \( w_i \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular.

(9e) Let \( L = \{ a^nb^l : n \leq l \leq 2n \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = a^nb^m \). Notice that \( w \in L \) (since \( m \leq m \leq 2m \)) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that \( y = a^k \) for some \( k \) with \( 1 \leq k \leq m \). Now let \( i = 2 \). Then \( w_i = w_2 = xy^2z = a^{m+k}b^m \). Then \( w_i \notin L \) since it is not the case that \( m + k \leq m \). This contradicts the Pumping Lemma. So \( L \) is not regular.

(9f) Let \( L = \{ a^nb^l : n \geq 100, l \leq 100 \} \). Then \( L \) is regular. Here is a regular expression for \( L \):

\[
a^{100}a^*(\lambda + b + bb + bbbb + bbbbb + \cdots + b^{98} + b^{99} + b^{100})
\]

(11) Let \( L_1 \) and \( L_2 \) be regular languages. Let \( L = \{ w : w \in L_1, w^R \in L_2 \} \). Then \( L \) is regular. To see this, just notice that \( L = L_1 \cap L_2^R \). Since the family of regular languages is closed under reversal and intersection, \( L \) is regular.
(13a) Let \( L = \{ uww^Rv : u, v, w \in \{a, b\}^+ \} \). Then \( L \) is regular. Let \( r \) be the following regular expression.
\[
(a + b)(a + b)^*(aa + bb)(a + b)^*.
\]

**Claim.** \( L = L(r) \).

**Proof.** First we will show that \( L \subseteq L(r) \). Let \( x \in L \). So then \( x = uww^Rv \) for some \( u, v, w \in \{a, b\}^+ \). Suppose the last symbol of \( w \) is \( a \). (If the last symbol of \( w \) is \( b \) the proof is similar.) Let us write \( w = ya \) with \( y \in \{a, b\}^+ \). Then we can write \( x = uyaay^Rv \). Now \( uy \in L((a + b)(a + b)^*) \) and \( y^Rv \in L((a + b)(a + b)^*) \) so \( x \in L(r) \).

Next we will show that \( L(r) \subseteq L \). Let \( x \in L(r) \). So then \( x = uaav \) or \( x = ubbv \) with \( u, v \in \{a, b\}^+ \). In either case we can write \( x = uww^Rv \) with \( u, v, w \in \{a, b\}^+ \). So \( x \in L \). \( \square \)

(13b) Let \( L = \{ uss^Rv : u, v, s \in \{a, b\}^+, |u| \geq |v| \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = (ab)^maa(ba)^m \). Notice that \( w \in L \) (with \( u = (ab)^m, s = a, v = (ba)^m \)) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Since \( |xy| \leq m \), we know that \( y \) is a substring of \((ab)^m\). Now let \( i = 0 \). Then \( w_0 = xz = raab(ba)^m \) for some \( r \) with \( |r| < |(ab)^m| = 2m \). I claim that \( w_0 \notin L \).

To see this, suppose towards a contradiction that \( w_0 \in L \). Then we can write \( w_0 = uss^Rv \) with \( u, v, s \in \{a, b\}^+ \) and \( |u| \geq |v| \). But also we know that \( w_0 = raab(ba)^m \). Since \( |r| < 2m \) but \( |u| \geq |v| \) we must have that \( ra \) is a prefix of \( u \). So \( ss^Rv \) is a substring of \( (ba)^m \). Now suppose the last symbol of \( s \) is \( a \). (If the last symbol of \( s \) is \( b \) the proof is similar.) Notice then that \( aa \) is a substring of \( ss^R \). But this is impossible because \( aa \) is not a substring of \( a(ba)^m \). This contradiction proves that \( w_0 \notin L \).

But this contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(14) Let \( L = \{ uu^Rv : u, v \in \{a, b\}^+ \} \). Then \( L \) is not regular.

**Proof.** This is a very difficult problem. It turns out that it is not possible to apply the Pumping Lemma directly to \( L \) in order to derive a contradiction. So I will use another strategy. Assume towards a contradiction that \( L \) is regular. Let \( r \) be the following regular expression: \((ab)^*(ab)(ba)^*b \). Let \( L_1 = L \cap L(r) \). If \( L \) is regular then so is \( L_1 \). We will apply the Pumping Lemma to \( L_1 \) to derive a contradiction. Notice that \( L_1 = \{ (ab)^*(ba)^t : t \geq s \geq 1 \} \). So assume that this \( L_1 \) is regular and we will derive a contradiction. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = (ab)^m(ba)^mb \). Notice that \( w \in L_1 \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Let us consider 4 possibilities for what \( y \) looks like:

**Case 1.** \( y \) starts with an \( a \) and ends with a \( b \).

So then \( y = (ab)^k \) for some \( k \) with \( 1 \leq k \leq m/2 \). In this case, let \( i = 2 \). Then \( w_1 = w_2 = xy^2z = (ab)^{m-k}(ba)^mb \). So \( w_1 \notin L \). But this contradicts the Pumping Lemma. So \( L_1 \) is not regular.

**Case 2.** \( y \) starts and ends with an \( a \).

In this case, let \( i = 2 \). Then \( w_1 = w_2 = xyyz \). Since \( y \) starts and ends with an \( a \), \( aa \) is a substring of \( yy \). But it is easy to see that \( aa \) is not a substring of any string in \( L_1 \). So \( w_2 \notin L_1 \). But this contradicts the Pumping Lemma. So \( L_1 \) is not regular.
Case 3. \( y \) starts and ends with an \( b \).
So then \( y = b(ab)^k \) for some \( k \) with \( 0 \leq k < m/2 \). Also \( x = (ab)^s a \) and \( z = (ab)^t (ba)^m b \) for some numbers \( s \) and \( t \) such that \( s + k + t + 1 = m \). In this case let \( i = 2 \). Then \( w_i = w_2 = xyyz = (ab)^s ab(ab)^k b(ab)^k (ab)^t (ba)^m b = (ab)^{s+k+t+1} b(ab)^{k+i} (ba)^m b \). Clearly \( w_2 \notin L(r) \) so \( w_2 \notin L_1 \). But this contradicts the Pumping Lemma. So \( L_1 \) is not regular.

Case 4. \( y \) starts with a \( b \) and ends with an \( a \).
So then \( y = b(ab)^k a \) for some \( k \) with \( 0 \leq k < m/2 \). Also \( x = (ab)^s a \) and \( z = b(ab)^t (ba)^m b \) for some numbers \( s \) and \( t \) such that \( s + k + t + 2 = m \). In this case let \( i = 2 \). Then \( w_i = w_2 = xyyz = (ab)^s ab(ab)^k ab(ab)^k ab(ab)^t (ba)^m b = (ab)^{s+k+t+3} (ba)^m b = (ab)^{m+1} (ba)^m b \). So \( w_2 \notin L_1 \). But this contradicts the Pumping Lemma. So \( L_1 \) is not regular.