Section 5.1

(4) Let \( L = \{ ab(bbaa)^n bba(ab)^n : n \geq 0 \} \). Then \( L \) is not regular.

**Proof.** Assume towards a contradiction that \( L \) is regular. Let \( m > 0 \) be given by the Pumping Lemma. Then let \( w = ab(bbaa)^m bba(ab)^m \). Notice that \( w \in L \) and \( |w| \geq m \). So let \( w = xyz \) be the decomposition of \( w \) given by the Pumping Lemma. Notice that there are many different possibilities for what \( y \) looks like and therefore it would seem as if this proof is going to be difficult. But in fact it will be easy. Let \( i = 2 \). I claim that \( w_2 \notin L \). To see this, I argue as follows. Notice that \( bba(ab)^m \) is a suffix of \( w \) and so it is also a suffix of \( w_2 \). But now notice that our string \( w \) is the one and only string in \( L \) that has \( bba(ab)^m \) as a suffix. Since \( w_2 \neq w \) and yet \( w_2 \) has \( bba(ab)^m \) as a suffix, \( w_2 \notin L \). This contradicts the Pumping Lemma. So \( L \) is not regular. \( \square \)

(6) (a) \[ S \to aSb | aaa | aa | a | \lambda \] (b) \[ S \to aSb | A | B | \lambda \]
\[ A \to aA | \lambda \]
\[ B \to bB | bb \]

(c) \[ S \to S_1 | S_2 \]

(d) \[ S \to aSbbB | \lambda \]
\[ B \to b | \lambda \]

As preparation for part (e) let us review exercise 14(b) from section 1.2 (Homework 1.) In that exercise you were asked for a grammar for the language \( L = \{ w \in \{ a, b \}^*: n_a(w) > n_b(w) \} \). My answer was the following grammar \( G \).

\[ S \to aS_1 | aS | S_1S \]
\[ S_1 \to S_1S_1 | \lambda | aS_1b | bSa \]

Let me explain why it is true that \( L(G) = L \). First notice that by Example 1.12 on page 23, \( S_1 \Rightarrow w \), iff \( n_a(w) = n_b(w) \). It is then clear that \( L(G) \subseteq L \), because the only way to eliminate the variable \( S \) is via the rule \( S \to aS_1 \). This adds one \( a \) and no \( b \).

So we must see that \( L \subseteq L(G) \). Now suppose \( w \in L \). We must see that \( S \Rightarrow^* w \). We argue by induction on \( |w| \). As the basis of the induction, the shortest string in \( L \) is the string \( a \), and it is true that \( S \Rightarrow^* a \). Now suppose that \( |w| \geq 2 \).

If \( w \) starts with a \( b \) then by an argument like the argument given in Example 1.12 on page 23, \( w \) must have a non-empty prefix \( w_1 \) such that \( n_a(w_1) = n_b(w_1) \). Now \( S_1 \Rightarrow^* w_1 \), and \( w = w_1w_2 \) with \( w_2 \) in \( L \). Since \( |w_2| < |w| \), by induction we know that \( S \Rightarrow^* w_2 \). Thus \( S \Rightarrow S_1S \Rightarrow^* w_1w_2 = w \).

So now suppose \( |w| \geq 2 \) and \( w = aw_1 \) for some string \( w_1 \). Then \( n_a(w_1) \geq n_b(w_1) \). If \( n_a(w_1) = n_b(w_1) \) then \( S_1 \Rightarrow^* w_1 \) and so \( S \Rightarrow aS_1 \Rightarrow^* aw_1 = w \). If \( n_a(w_1) > n_b(w_1) \) then \( w_1 \in L \), and since \( |w_1| < |w| \) by induction we know that \( S \Rightarrow^* w_1 \). So \( S \Rightarrow aS \Rightarrow^* aw_1 = w \).
Given the previous discussion, I hope you will understand the solution to part (e) below.

\[
\begin{align*}
S & \rightarrow S_a \mid S_b \\
\text{(e)} \quad S_a & \rightarrow aS_1 \mid aS_a \mid S_1S_a \\
S_b & \rightarrow bS_1 \mid bS_b \mid S_1S_b \\
S_1 & \rightarrow S_1S_1 \mid \lambda \mid aS_1b \mid bS_1a
\end{align*}
\]

Now I explain my solution to part (f). Let \(G\) be the grammar in part (f) above and let \(L\) the language given in the textbook exercise. I will show that \(L(G) = L\). By induction on the length of a derivation of a sentential form \(u\), we can see that every sentential form \(u\) has the property that if \(v\) is a prefix of \(u\) then \(n_a(v) \geq n_a(u)\). This proves that \(L(G) \subseteq L\). So now we must see that \(L \subseteq L(g)\). We will prove by induction on the length of \(w\) that if \(w \in L\) then \(S \Rightarrow^* w\). As the basis step of the induction, if \(w = \lambda\) then indeed \(S \Rightarrow^* w\). So now suppose that \(w \notin L\). Then there is some prefix \(v\) of \(w\) such that \(n_a(v) > n_a(u)\). Let \(v\) be the shortest such prefix. Notice then that the final symbol of \(v\) is \(a\). So let us write \(w = aw_1\). If \(w_1 \in L\) then since \(|w_1| < |w|\) we have by induction that \(S \Rightarrow^* w_1\) and so \(S \Rightarrow aS \Rightarrow^* aw_1 = w\). So now suppose that \(w_1 \notin L\). Then there is some prefix \(v\) of \(w_1\) such that \(n_b(v) > n_a(u)\). Let \(v\) be the shortest such prefix. Notice then that the final symbol of \(v\) is \(b\) and that and \(n_b(v) = n_a(v) + 1\). Let us write \(v = xb\). Notice that \(x \in L\) and \(n_a(x) = n_b(x)\). Also let us write \(w_1 = vy = xby\). Notice then that \(w = aw_1 = axby\). Now \(n_a(axb) = n_b(axb)\) and so we must have that \(y \in L\). Since \(x\) and \(y\) are in \(L\) and \(|x|, |y| < |w|\), we have by induction that \(S \Rightarrow^* x\) and \(S \Rightarrow^* y\). It then follows that \(S \Rightarrow aSbS \Rightarrow^* axby = w\).
Section 7.1

\( z = 0, \ F = \{q_3\} \)

\[
\begin{align*}
\delta(q_0, \lambda, 0) &= \{(q_3, \lambda)\} \\
\delta(q_0, a, 0) &= \{(q_1, 110)\} \\
\delta(q_1, a, 1) &= \{(q_1, 111)\} \\
\delta(q_1, b, 1) &= \{(q_2, \lambda)\} \\
\delta(q_2, b, 1) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, 0) &= \{(q_3, \lambda)\}
\end{align*}
\]

\( \delta(q_0, a, z) = \{(q_0, az)\} \)

\( \delta(q_0, b, z) = \{(q_0, bz)\} \)

\( \delta(q_0, a, a) = \{(q_0, aa)\} \)

\( \delta(q_0, b, a) = \{(q_0, ba)\} \)

\( \delta(q_0, a, b) = \{(q_0, ab)\} \)

\( \delta(q_0, b, b) = \{(q_0, bb)\} \)

\( \delta(q_0, c, z) = \{(q_1, z)\} \)

\( \delta(q_0, c, a) = \{(q_1, a)\} \)

\( \delta(q_0, c, c) = \{(q_1, b)\} \)

\( \delta(q_1, a, a) = \{(q_1, \lambda)\} \)

\( \delta(q_1, b, b) = \{(q_1, \lambda)\} \)

\( \delta(q_1, \lambda, z) = \{(q_2, z)\} \)

\( z = 0, \ F = \{q_3\} \)

\( \delta(q_0, a, 0) = \{(q_0, 10)\} \)

\( \delta(q_0, a, 1) = \{(q_0, 11)\} \)

\( \delta(q_0, b, 0) = \{(q_1, 10)\} \)

\( \delta(q_0, b, 1) = \{(q_1, 11)\} \)

\( \delta(q_0, c, 1) = \{(q_2, \lambda)\} \)

\( \delta(q_2, c, 1) = \{(q_2, \lambda)\} \)

\( \delta(q_2, \lambda, 0) = \{(q_3, \lambda)\} \)

\( F = \{q_2\} \)

\( \delta(q_0, b, z) = \{(q_0, 0z)\} \)

\( \delta(q_0, b, 0) = \{(q_0, 00)\} \)

\( \delta(q_0, a, z) = \{(q_1, \lambda)\} \)

\( \delta(q_0, z, 0) = \{(q_1, 0)\} \)

\( \delta(q_1, b, 0) = \{(q_1, 00)\} \)

\( \delta(q_1, b, z) = \{(q_1, 0z)\} \)

\( \delta(q_1, a, 0) = \{(q_1, 1\lambda)\} \)

\( \delta(q_1, a, 1) = \{(q_1, 11)\} \)

\( \delta(q_1, a, z) = \{(q_1, 1z)\} \)

\( \delta(q_1, a, 0) = \{(q_1, \lambda)\} \)

\( \delta(q_0, a, z) = \{(q_0, 1z)\} \)

\( \delta(q_0, a, 1) = \{(q_0, 11)\} \)

\( \delta(q_0, a, 0) = \{(q_0, \lambda)\} \)

\( \delta(q_0, b, z) = \{(q_0, 0z)\} \)

\( \delta(q_0, b, 0) = \{(q_0, 00)\} \)

\( \delta(q_0, b, 1) = \{(q_0, 1\lambda)\} \)

\( \delta(q_0, c, 0) = \{(q_0, 00)\} \)

\( \delta(q_0, c, 1) = \{(q_0, \lambda)\} \)

\( \delta(q_0, \lambda, z) = \{(q_2, z)\} \)

\( \delta(q_1, \lambda, 1) = \{(q_0, \lambda)\} \)

\( \delta(q_1, \lambda, 0) = \{(q_0, 00)\} \)

\( \delta(q_1, \lambda, z) = \{(q_0, 0z)\} \)

\( \delta(q_0, \lambda, z) = \{(q_1, z)\} \)
(5) \( F = \{q_f\} \)

\[ \delta(q_0, \lambda, z) = \{(q_1, z), (q_4, z)\} \]
\[ \delta(q_1, a, z) = \{(q_2, z)\} \]
\[ \delta(q_2, a, z) = \{(q_2, z), (q_2, 1z)\} \]
\[ \delta(q_2, b, 1) = \{(q_3, \lambda)\} \]
\[ \delta(q_3, b, 1) = \{(q_3, \lambda)\} \]
\[ \delta(q_3, \lambda, z) = \{(q_f, z)\} \]
\[ \delta(q_4, a, z) = \{(q_4, 1z)\} \]
\[ \delta(q_4, a, 1) = \{(q_4, 11)\} \]
\[ \delta(q_4, b, z) = \{(q_5, z)\} \]
\[ \delta(q_4, b, 1) = \{(q_5, 1)\} \]
\[ \delta(q_5, b, 1) = \{(q_5, 1), (q_5, \lambda)\} \]
\[ \delta(q_5, \lambda, z) = \{(q_f, z)\} \]

(10) Note that there is never more than one symbol on the stack. So this npda is equivalent to an nfa. The language accepted is given by the regular expression \( a + abb^*a \).

(11) Every string is accepted. The language is \( \{a, b\}^* \).

(12) \( L(\lambda + a + abb^* + abb^*a) \).

(14) \( \{a\} \).