Theory of Algorithms. Spring 2000. Homework Solutions 8.

Section 7.2 The Equivalence of NPDAs and Context-Free Grammars.

(1) Let G be the following context-free grammar, with $V = \{S, B, C\}$, $T = \{a, b, c\}$.

$$S \to aSbbB \mid C \mid \lambda$$
$$B \to b \mid \lambda$$
$$C \to Cc \mid c$$

(a) Find an npda M such that $L(M) = L(G)$.

SOLUTION: $F = \{q_f\}$

$\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$
$\delta(q_1, a, a) = \{(q_1, \lambda)\}$ $\quad$ $\delta(q_1, \lambda, S) = \{(q_1, SaSbbB), (q_1, C), (q_1, \lambda)\}$
$\delta(q_1, b, b) = \{(q_1, \lambda)\}$ $\quad$ $\delta(q_1, \lambda, B) = \{(q_1, b), (q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_f, \lambda)\}$ $\quad$ $\delta(q_1, \lambda, C) = \{(q_1, Cc), (q_1, c)\}$

(b) Give a left-most derivation from G of the string: $aaccbbbbb$.

SOLUTION:

$S \Rightarrow aSbbB \Rightarrow aaSbbBbbB \Rightarrow aaCbbBbbB \Rightarrow aaCcbbBbbB \Rightarrow aaccbbBbbB \Rightarrow aaccbbbbbB \Rightarrow aaccbbbbb$

(c) Give the corresponding sequence of instantaneous descriptions for M.

SOLUTION:

$(q_0, aaccbbbbb, z) \vdash (q_1, aaccbbbbb, Sz) \vdash (q_1, aaccbbbbb, aSbbBz) \vdash (q_1, accbbbbb, SbbBz) \vdash$
$(q_1, accbbbbb, aSbbBbbBz) \vdash (q_1, ccbbbbb, SbbBbbBz) \vdash (q_1, ccbbbbb, CbbBbbBz) \vdash$
$(q_1, ccbbbbb, CcbbBbbBz) \vdash (q_1, ccbbbbb, ccbbBbbBz) \vdash (q_1, cbbbbb, cbbBbbBz) \vdash (q_1, bbbbb, bbBbbBz) \vdash$
$(q_1, bbbb, bBbbBz) \vdash (q_1, bbb, BbbBz) \vdash (q_1, bbb, bbbBz) \vdash (q_1, bb, bbBz) \vdash (q_1, b, bBz) \vdash (q_1, \lambda, Bz) \vdash$
$(q_1, \lambda, z) \vdash (q_f, \lambda, \lambda) \vdash$ **"ACCEPT"**

(2) Let M be the following npda, with $\Gamma = \{S, 0, 1, z\}$, $F = \{q_f\}$.

$\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$
$\delta(q_1, a, S) = \{(q_1, 11S)\}$
$\delta(q_1, a, 1) = \{(q_1, 11)\}$
$\delta(q_1, a, 0) = \{(q_1, \lambda), (q_1, 010)\}$
$\delta(q_1, b, 1) = \{(q_1, \lambda)\}$
$\delta(q_1, b, S) = \{(q_1, \lambda), (q_1, 0S)\}$
$\delta(q_1, \lambda, 0) = \{(q_1, 000), (q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_f, \lambda)\}$

Find a context-free grammar G such that $L(G) = L(M)$.

SOLUTION: $V = \{S, 0, 1, z\}$, $T = \{a, b\}$.

$$S \to a11S \mid b \mid b0S$$
$$1 \to a11 \mid b$$
$$0 \to a \mid a0101 \mid 000 \mid \lambda$$

(3) (OPTIONAL) Let M' be the following npda, with
$Q' = \{q'_0, q'_1, q'_2\}$, $F' = \{q'_1, q'_2\}$, $\Sigma' = \{a, b\}$, and $\Gamma' = \{z', a, b\}$.
$\delta'(q'_0, a, z') = \{(q'_0, az'), (q'_2, \lambda)\}$
$\delta'(q'_0, a, a) = \{(q'_0, aa), (q'_1, \lambda)\}$
$\delta'(q'_0, b, a) = \{(q'_0, ba)\}$
$\delta'(q'_1, b, b) = \{(q'_1, \lambda)\}$
$\delta'(q'_1, a, a) = \{(q'_1, \lambda)\}$
$\delta'(q'_2, a, a) = \{(q'_2, a)\}$
Find an npda M such that $L(M) = L(M')$ and M is in pre-standard form.

SOLUTION. $Q = \{q_0, q_f, q^*, q'_0, q'_1, q'_2\}$, $F = \{q_f\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, z', a, b\}$.

$\delta(q'_0, a, z') = \{(q'_0, az'), (q'_2, \lambda)\}$
$\delta(q'_0, a, a) = \{(q'_0, aa), (q'_1, \lambda)\}$
$\delta(q'_0, b, a) = \{(q'_0, ba)\}$
$\delta(q'_1, b, b) = \{(q'_1, \lambda)\}$
$\delta(q'_1, a, a) = \{(q'_1, \lambda)\}$
$\delta(q'_2, a, a) = \{(q'_2, a)\}$

$\delta(q^*, \lambda, a) = \{(q^*, \lambda)\}$
$\delta(q^*, \lambda, b) = \{(q^*, \lambda)\}$
$\delta(q^*, \lambda, z') = \{(q^*, \lambda)\}$
$\delta(q^*, \lambda, z) = \{(q_f, \lambda)\}$

$\delta(q_0, \lambda, z) = \{(q'_0, z'z)\}$
$\delta(q'_1, \lambda, z') = \{(q^*, z')\}$
$\delta(q'_1, \lambda, a) = \{(q^*, a)\}$
$\delta(q'_1, \lambda, b) = \{(q^*, b)\}$
$\delta(q'_1, \lambda, z) = \{(q^*, z)\}$
$\delta(q'_2, \lambda, z') = \{(q^*, z')\}$
$\delta(q'_2, \lambda, a) = \{(q^*, a)\}$
$\delta(q'_2, \lambda, b) = \{(q^*, b)\}$
$\delta(q'_2, \lambda, z) = \{(q^*, z)\}$

(4) Let M be the following npda with $F = \{q_f\}$.

$\delta(q_0, a, z) = \{(q_0, az)\}$
$\delta(q_0, a, a) = \{(q_0, aa)\}$
$\delta(q_0, a, b) = \{(q_0, ab)\}$
$\delta(q_0, b, z) = \{(q_0, bz)\}$
$\delta(q_0, b, a) = \{(q_0, ba)\}$
$\delta(q_0, b, b) = \{(q_0, bb)\}$

$\delta(q_1, a, a) = \{(q_1, \lambda)\}$
$\delta(q_1, b, b) = \{(q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_f, \lambda)\}$

$\delta(q_0, \lambda, a) = \{(q_1, a)\}$
$\delta(q_0, \lambda, b) = \{(q_1, b)\}$

(a) Find an npda $\bar{M}$ such that $L(\bar{M}) = L(M)$, and $\bar{M}$ is in standard reduced form. Use the clause-template notation that I used in the example in the notes.

SOLUTION. $\bar{M} = (\bar{Q}, \bar{\Sigma}, \bar{\Gamma}, \bar{\delta}, \bar{q}_0, \bar{z}, \bar{F})$ with $\bar{Q} = \{\bar{q}_0, \bar{q}_1, \bar{q}_f\}$, $\bar{F} = \{\bar{q}_f\}$, $\bar{\Sigma} = \{a, b\}$. We also set $\bar{\Gamma} = (\{q_0, q_1, q_f\} \times \{a, b, z\} \times \{q_0, q_1, q_f\}) \cup \{\bar{z}\}$. Since $\bar{M}$ is going to be in standard-reduced form, we must choose one of the stack symbols to be $\bar{S}$. We will use $\bar{S} = (q_0, z, q_f)$.

Now we define $\bar{\delta}$. Since $\bar{M}$ will be in standard-reduced form we have the following two clauses:

(1) $\bar{\delta}(\bar{q}_0, \lambda, \bar{z}) = \big\{(\bar{q}_1, \bar{S}\bar{z})\big\}$ **(2)** $\bar{\delta}(\bar{q}_1, \lambda, \bar{z}) = \{(\bar{q}_f, \lambda)\}$.

Now for the rest of $\bar{\delta}$. First the clauses that define when $\bar{M}$ can *decrease* the size of its stack:

These clauses for $\bar{M}$	come from these clauses for M.
(3) $\bar{\delta}\big(\bar{q}_1, a, (q_1, a, q_1)\big) = \{(\bar{q}_1, \lambda)\}$	$\delta(q_1, a, a) = \{(q_1, \lambda)\}$
(4) $\bar{\delta}\big(\bar{q}_1, b, (q_1, b, q_1)\big) = \{(\bar{q}_1, \lambda)\}$	$\delta(q_1, b, b) = \{(q_1, \lambda)\}$
(5) $\bar{\delta}\big(\bar{q}_1, \lambda, (q_1, z, q_f)\big) = \{(\bar{q}_1, \lambda)\}$	$\delta(q_1, \lambda, z) = \{(q_f, \lambda)\}$

Next the clauses that define when $\bar{M}$ can *increase* the size of its stack.

In M we have the clauses:
$\delta(q_0, a, z) = \{(q_0, az)\}$.
$\delta(q_0, a, a) = \{(q_0, aa)\}$.
$\delta(q_0, a, b) = \{(q_0, ab)\}$.
$\delta(q_0, b, z) = \{(q_0, bz)\}$.
$\delta(q_0, b, a) = \{(q_0, ba)\}$.
$\delta(q_0, b, b) = \{(q_0, bb)\}$.

These yields the following clause templates in $\bar{M}$:

(6) $$\bar{\delta}\big(\bar{q}_1, a, (q_0, z, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, a, q_y)(q_y, z, q_x)\Big)\Big\}.$$

(7) $$\bar{\delta}\big(\bar{q}_1, a, (q_0, a, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, a, q_y)(q_y, a, q_x)\Big)\Big\}.$$

(8) $$\bar{\delta}\big(\bar{q}_1, a, (q_0, b, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, a, q_y)(q_y, b, q_x)\Big)\Big\}.$$

(9) $$\bar{\delta}\big(\bar{q}_1, b, (q_0, z, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, b, q_y)(q_y, z, q_x)\Big)\Big\}.$$

(10) $$\bar{\delta}\big(\bar{q}_1, b, (q_0, a, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, b, q_y)(q_y, a, q_x)\Big)\Big\}.$$

(11) $$\bar{\delta}\big(\bar{q}_1, b, (q_0, b, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_0, b, q_y)(q_y, b, q_x)\Big)\Big\}.$$

Finally there are two clauses that leave the size of the stack the same. I did not cover this in the example in the notes. It works exactly the same as the clauses that increase the size of the stack, except that in the clause template we only have the variable q_x.

In M we have the clauses: $\delta(q_0, \lambda, a) = \{(q_1, a)\}$ and $\delta(q_0, \lambda, b) = \{(q_1, b)\}$.

These yields the following clause templates in $\bar{M}$:

(12) $$\bar{\delta}\big(\bar{q}_1, \lambda, (q_0, a, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_1, a, q_x)\Big)\Big\}.$$

(12) $$\bar{\delta}\big(\bar{q}_1, \lambda, (q_0, b, q_x)\big) = \Big\{ \Big(\bar{q}_1, (q_1, b, q_x)\Big)\Big\}.$$

(b) Give the sequences of instantaneous descriptions for M and $\bar{M}$ that show that M and $\bar{M}$ accept the string: *abaaba*.

SOLUTION:

In M: $(q_0, abaaba, z) \vdash (q_0, baaba, az) \vdash (q_0, aaba, baz) \vdash (q_0, aba, abaz) \vdash (q_1, aba, abaz) \vdash (q_1, ba, baz) \vdash (q_1, a, az) \vdash (q_1, \lambda, z) \vdash (q_f, \lambda, \lambda) \vdash$ **"ACCEPT"**

In $\bar{M}$: $(\bar{q}_0, abaaba, \bar{z}) \vdash (\bar{q}_1, abaaba, (q_0, z, q_f)\bar{z}) \vdash (\bar{q}_1, baaba, (q_0, a, q_1)(q_1, z, q_f)\bar{z}) \vdash$
$(\bar{q}_1, aaba, (q_0, b, q_1)(q_1, a, q_1)(q_1, z, q_f)\bar{z}) \vdash (\bar{q}_1, aba, (q_0, a, q_1)(q_1, b, q_1)(q_1, a, q_1)(q_1, z, q_f)\bar{z}) \vdash$
$(\bar{q}_1, aba, (q_1, a, q_1)(q_1, b, q_1)(q_1, a, q_1)(q_1, z, q_f)\bar{z}) \vdash$
$(\bar{q}_1, ba, (q_1, b, q_1)(q_1, a, q_1)(q_1, z, q_f)\bar{z}) \vdash (\bar{q}_1, a, (q_1, a, q_1)(q_1, z, q_f)\bar{z}) \vdash (\bar{q}_1, \lambda, (q_1, z, q_f)\bar{z}) \vdash (\bar{q}_1, \lambda, \bar{z}) \vdash$
$(\bar{q}_f, \lambda, \lambda) \vdash$ **"ACCEPT"**