a) Checking \( f(x) = \sqrt{x(1-x)} \) on \([0,1]\) for applicability of Rolle. Just in case \( D: x(1-x) > 0 \)

\[ a + \frac{1}{b} + \frac{1}{c} \]

Part B is in dom: \( \frac{1}{2}(1-\frac{1}{2}) > 0. \ D: [0,1]. \)

Contin on \([0,1]\)? Yes (no denominator to make discontinuities)

\[ f'(x) = \frac{1-2x}{2\sqrt{x-x^2}} \]

inde ned if \( x = 0 \) or \( x = 1 \) but these are the endpoints. Must be defined on \((0,1)\)
and it is.

Find \( x \) int: \( y = \sqrt{x(1-x)} \); let \( y = 0 \) \( \sqrt{x(1-x)} = 0 \Rightarrow \)
\( x = 0, 1. \)

Funct \( f(x) \) is good!

Then we apply Rolle to prove that somewhere on \((0,1)\) there is a point (called \( C \)) so that \( f'(x) = 0 \)

\[ \frac{1-2x}{2\sqrt{x-x^2}} = 0 \Rightarrow 1-2x = 0 \Rightarrow 1 = 2x \Rightarrow \frac{1}{2} = x \]

\[ 0 < \frac{1}{2} < 1 \] Worked!

b) \( f(x) = \frac{x^2-1}{x-2} \) Disc. at \( x = 2 \); Contin on \([-1,1]\)? Yes.

\[ f'(x) = \frac{2x(x-2)-(x^2-1)}{(x-2)^2} = \frac{2x^2-4x-x^2+1}{(x-2)^2} = \frac{x^2-4x+1}{(x-2)^2} \]

Defined at all points except \( x = 2 \), Defined on \((-1,1)\).

Find \( x \) int: \( \frac{x^2-1}{x-2} = 0 \Rightarrow x^2-1 = 0 \Rightarrow x^2 = 1, \ x = \pm 1 \)

The function is good!

Apply Rolle Theorem, i.e. prove that on \((-1,1)\) there is a point \( C \) where \( f'(x) = 0 \):

\[ \frac{x^2-4x+1}{(x-2)^2} = 0 \Rightarrow x^2-4x+1 = 0, \ x = \frac{4 \pm \sqrt{16-4 \times 1}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 3.46}{2} \]

\( x_1 = 7.46 \approx 3.73 \) (out of interval \((-1,1)\)).

And \( x_2 = 0.27 \), this is the place \( C \).

\[ 0.27 \] Worked.

c) \( f(x) = x - \frac{1}{x^2} \). Continuous on \([-2,2]\)?

No! Not continuous at \( x = 0 \). Discard the function.
(d) \( f(x) = \frac{1}{2}x - \sqrt{x} \) on \([0,4]\)

Contin. on \([0,4]\)? Yes (no denom. that would contain \(x\) to make it discontinuous).

\( f'(x) = \frac{1}{2} - \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{x}} \right) \). Not defined at \(x=0\), which is an endpoint of \([0,4]\). Must be defined on \((0,4)\) - and it is.

Find \(x\) intercepts: \(\frac{1}{2}x - \sqrt{x} = 0\), \(\frac{1}{4}x = \sqrt{x}\), Square:

\(\frac{1}{4}x^2 = x\); \(\frac{1}{4}x^2 - x = 0\), \(x(\frac{1}{4}x - 1) = 0\), \(x=0\), \(x=4\)

The function is good.

Then look for a place where \(f'(x) = 0\). Demonstrate that this would happen on \((0,4)\):

\(\frac{1}{2} \left( 1 - \frac{1}{\sqrt{x}} \right) = 0\), \(1 - \frac{1}{\sqrt{x}} = 0\), \(\frac{1}{\sqrt{x}} = 1\). Cross-multiply:

\(\sqrt{x} = 1\). Square: \(x=1\) 

0 1 4 

Rolle worked.

Mean Value. (we don’t need the requirement that \(a\) and \(b\) are \(x\) intercept. But we want to make sure that \(f(x)\) is contin. on \([a,b]\) and that \(f'(x)\) is defined on \((a,b)\). If this is not so then Mean Value theorem does not apply.

a) \( f(x) = x^2 + 2x - 1 \) on \([0,1]\) 

Contin on \([0,1]\) - Yes, no denominator to make it discontinuous.

\( f'(x) = 2x + 2 \), defined everywhere including \((0,1)\).

The function is good.

Now let us calculate the slopes of a sec. line, tang. line, and equate the two.

for sec. line: \(M = \frac{f(1) - f(0)}{1 - 0} = \frac{(1^2 + 2\cdot1 - 1) - (-1)}{1} = \frac{3 - 3}{1} = \frac{1}{2} \)

\( f'(x) = 2x + 2 \); Equation: \(2x + 2 = 3\), \(2x = 1\), \(x = \frac{1}{2}\)

0 \(\frac{1}{2}\) 1 

The theorem worked!
b) \( f(x) = x + \frac{1}{x} \) on \([-1, 1]\). Contin on \([-1, 1]\)? No! Discontin at \(x = 0\). Discard!

c) \( f(x) = x^{1/2} \) on \([\frac{1}{2}, 2]\)

Contin on \([\frac{1}{2}, 2]\), \( f'(x) = 1 - \frac{1}{x^2} \), Not defined at \(x = 0\), but most certainly defined on \((\frac{1}{2}, 2)\).

M sec. line: \( \frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = \frac{(2^{\frac{1}{2}}) - (\frac{1}{2^{\frac{1}{2}}})}{\frac{3}{2}} = 0 = 0 \)

\( f'(x) = 1 - \frac{1}{x^2} \); Equate: \( 1 - \frac{1}{x^2} = 0 \); \( 1 = \frac{1}{x^2} \); \( x^2 = 1 \), \( x = 1, -1 \)

\( \frac{1}{2} \) \( 1 \) \( 2 \)

If you look carefully, this was a case for Rolle's theorem.

Rolle is a particular case of Mean Value. Secant line goes along the x axis; tan line is parallel to sec line, and this happens at \(x = 1\) on \([\frac{1}{2}, 2]\).

d) \( f(x) = \sqrt{x^2} \) or \( x^{2/3} \) on \([-1, 8]\)

Contin on \([-1, 8]\)? Yes (no denominator to make it discontinuous)

\( f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3 \sqrt[3]{x}} \). Not defined at \(x = 0\), within \((-1, 8)\). Discard the function!

e) \( f(x) = \cos x \) on \([-\frac{\pi}{2}, 0]\)

Contin on \([-\frac{\pi}{2}, 0]\)? Yes;

\( f'(x) = -\sin x \); defined on \((-\frac{\pi}{2}, 0)\)? Yes

M sec. line = \( \frac{\cos 0 - \cos(-\frac{\pi}{2})}{0 - (-\frac{\pi}{2})} = \frac{1 - 0}{\frac{\pi}{2}} = \frac{2}{\pi} \); Equate.
\[ f'(x) \text{ and } \frac{2}{\pi} : \quad -\sin x = \frac{2}{\pi} ; \quad -\sin x = \frac{2}{3.14} = 0.6369 \]

\[ \sin x = -0.6369. \] Apply operation \( \sin^{-1} \) to the left and right side, like in \( +\text{rig} : x = -39.56^\circ \text{ or } -40^\circ \)

\[ -40^\circ \]

\[ -90^\circ \]

\[ \text{The theorem worked!} \]

\[ f(x) = \cot x \text{ on } \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \]

Contin on \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \) - No.

Discard the function.

\[ \underline{\text{FOR FUN!}} \text{ Ignore that } f(x) \text{ was discarded. Go ahead and try to apply the Mean-Val theorem:} \]

The sec. line (see graph) goes through the marked dots and has the slope of 0.
Then \( f'(x) = -\csc^2 x \) must equal 0 on the interval. But look at the graph of \( \csc x \).

\[ y = \csc x. \rightarrow \csc x \text{ never } = 0 \]

\[ \text{since the } y \text{ coord. is never } 0. \]

The theorem did not work!
It was not possible to find a tangent line (through \( f'(x) \)) with the same slope as the secant line. Why? because the function \( f(x) = \cot x \) does not meet the conditions of Mean-Val theorem and was rightly discarded.
Part 3

1) \[ \int (x + x^4) \, dx = \frac{1}{2} x^2 + \frac{1}{8} x^5 + c \]

2) \[ \int \left( \frac{1}{x^3} - \frac{2x^3}{x^3} \right) \, dx = \int (x^{-3} - 2) \, dx = \frac{1}{2} x^{-2} - 2x + c = \frac{1}{2} x^{-2} - 2x + c \]

3) \[ \int \left( \frac{1}{2} \cdot \frac{1}{y} + e^y \right) \, dy = \frac{1}{2} \ln |y| + e^y + c \]

4) \text{Ans: } \ln |\theta| - 2e^\theta - \cot \theta + c \]

5) \[ \int \left( \frac{1}{4} \cdot \frac{1}{x} + \frac{1}{3} X^{-2/3} + \frac{2}{3} x^2 - \sqrt{5} \cdot X^{3/2} + \frac{1}{\sqrt{5}} \right) \, dx = \frac{1}{4} \ln |x| + \frac{1}{3} \cdot 3X^{1/3} + \frac{2}{3} \cdot \frac{1}{2} x^{-1} - \sqrt{5} \cdot \frac{3}{5} x^{5/2} + \frac{1}{\sqrt{5}} \cdot x + c \]

6) \text{Ans: } -4\cos x - 2\sin x + c \]

7) \text{Ans: } -\csc x + c \]

Part 4

1) \[ \int \left( 4x^3 + 5 \right)^{-\frac{1}{2}} \cdot x^2 \, dx \]

\[ \text{Let } u = 4x^3 + 5 \]

\[ \frac{du}{dx} = 12x^2 \]

\[ \int u^{-\frac{1}{2}} \cdot \frac{1}{12} \, du = \frac{1}{12} \cdot 2 \cdot u^{\frac{1}{2}} + c = \frac{1}{6} \sqrt{4x^3 + 5} + c \]

2) \[ \int \frac{x^2}{4x^3 + 5} \, dx = \frac{1}{12} \int \frac{12x^2}{4x^3 + 5} \, dx = \frac{1}{12} \ln |4x^3 + 5| + c \]

3) \[ \int (1+y^2)^{\frac{1}{2}} \cdot y \, dy \]

\[ \text{Let } u = 1+y^2 \]

\[ \frac{du}{dy} = 2y \]

\[ \int u^{\frac{1}{2}} \cdot \frac{1}{2} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + c = \frac{1}{3} \sqrt{(1+y^2)^3} + c \]

5) \[ \int e^{-3x} \, dx \]

\[ \text{Let } u = -3x \]

\[ \frac{du}{dx} = -3 \]

\[ \int e^u \cdot \frac{-1}{3} \, du = \frac{-1}{2} e^u + c \]

\[ \int e^u \cdot \frac{-1}{3} \, du = \frac{-1}{2} e^{-3x} + c \]
4) \( \int (1 + \sqrt{x})^{4} \cdot x^{-1/2} \, dx \)

\[
\int u^4 \cdot 2 \, du = 2 \cdot \frac{1}{5} u^5 + C = \frac{2}{5} (1 + \sqrt{x})^{5} + C
\]

5) \( \rightarrow \) done.

6) \( \int \cos \left( \frac{x}{2} \right) \, dx \)

\[
\int \cos x \cdot 2 \, du = 2 \sin \left( \frac{x}{2} \right) + C
\]

7) \( \int (x^2 + 7x + 3)^{-4} \cdot (2x + 7) \, dx \)

\[
\int u^{-4} \cdot du = -\frac{1}{3} u^{-3} + C = \frac{-1}{3(x^2 + 7x + 3)^3} + C
\]

8) On top: exact due of the denominator—therefore resemblance to \( \int \frac{1}{x} \, dx \) and a ln function for the ans.

ANS: \( \ln |x^2 + 7x + 3| + C \)

9) \( \int (2 - x)^{-1/2} \, dx \)

\[
\int u^{-1/2} \cdot -\frac{1}{2} \, du = -\frac{1}{2} \cdot 2 u^{1/2} + C = -\sqrt{2-x} + C
\]

10) \( \int (3x - 1)^5 \, dx \)

\[
\int u^5 \cdot \frac{1}{3} \, du = \frac{1}{3} \cdot \frac{1}{6} u^6 + C = \frac{1}{18} (3x-1)^6 + C
\]

11) \( \int \frac{1}{3x - 1} \, dx \)

\[
\int \frac{1}{3} \int \frac{3}{3x - 1} \, dx = \frac{1}{3} \ln |3x - 1| + C
\]

12) \( \int (3x - 1)^{-1/2} \, dx \)

\[
\int u^{-1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \cdot \frac{2}{3} u^{1/2} + C = \frac{2}{3} \sqrt{3x - 1} + C
\]
13) \( \int \cos^2 \left( \sqrt{x} \right) \cdot \frac{x^{-1/2}}{2} \, dx \)

\( \text{Let } u = \sqrt{x} \)
\( \frac{du}{2} = x^{-1/2} \, dx \)

\( \int \csc^2 u \cdot 2 \, du = -2 \cot u + C = -2 \cot \sqrt{x} + C \)

14) \( \int \frac{\cos x}{2 + \sin x} \, dx = \ln |2 + \sin x| + C \) (similar to \( \int \frac{1}{x} \, dx \), top = exact deriv of bottom). Also:
Ans \( \ln (2 + \sin x) + C \) is also correct; here \( 2 + \sin x \) is always pos since the worst-case scenario when \( \sin x = -1 \). No need to "eliminate negatives" before they enter the logar. function

15) \( \int [2 + \sin(3x)]^{-1/2} \cos(3x) \, dx \)

\( \text{Let } u = 2 + \sin(3x) \)
\( du = 3 \cos(3x) \, dx \)

\( \int u^{-1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \cdot 2u^{1/2} + C = \sqrt{2 + \sin(3x)} + C \)

16) \( \int e^{\tan x} \cdot \sec^2 x \, dx \)

\( \text{Let } u = \tan x \)
\( du = \sec^2 x \, dx \)

\( \int e^u \, du = e^u + C = e^{\tan x} + C \)

17) \( \int e^{\sqrt{x}} \cdot \left( \frac{x^{-1/2}}{2} \right) \, dx \)

\( \text{Let } u = \sqrt{x} \)
\( du = \frac{1}{2} x^{-1/2} \, dx \)

\( \int e^u \cdot 2 \, du = 2 e^u + C = 2 e^{\sqrt{x}} + C \)

18) \( \int \sin \left( \frac{1}{x} \right) \cdot x^{-2} \, dx \)

\( \text{Let } u = \frac{1}{x} \)
\( du = -x^{-2} \, dx \)

\( \int -u \, du = \int -\sin \, du = \cos u + C \)

19) \( \int \frac{\sin x}{\cos x} \, dx = -\int -\frac{\sin x}{\cos x} \, d(x) = -\ln |\cos x| + C \)

Ans \( \ln \left| \frac{1}{\cos x} \right| + C \) is also correct. Check:
\( \ln \left| \frac{1}{\cos x} \right| + C = \ln 1 - \ln |\cos x| + C \)
2b) \( \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x| + c \).

2) \( \int \frac{\sin(5x)}{\cos(5x)} \, dx \). The top should be \(-\sin(5x) \cdot \frac{1}{5}\) if we wish to justify the ans as a \(\ln\) function. Missing on top: \(-\frac{1}{5}\). Put it in and compensate:

\[
\frac{-1}{5} \int \frac{-5\sin(5x)}{\cos(5x)} \, dx = -\frac{1}{5} \ln |\cos(5x)| + c.
\]

2a) \( \int \frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})} \, dx \). The top should be \(\cos(\frac{x}{2}) \cdot \frac{1}{a}\).

Put the missing \(\frac{1}{2}\) on top and compensate with \(\frac{1}{2}\) in front of \(\sin\):

\[
2 \int \frac{\cos(\frac{x}{2}) \cdot \frac{1}{2}}{\sin(\frac{x}{2})} \, dx = 2 \ln |\sin(\frac{x}{2})| + C.
\]