INTEGRATION BY SUBSTITUTION

In our course, any integral which cannot be found using a direct formula or manipulated by using algebra and/or trig into such an integral will require the substitution technique to compute. The purpose of integration by substitution is to reverse the chain rule. Since \( \left( f(g(x)) \right)' = f'(g(x))g'(x) \), we have that \( \int f'(g(x))g'(x) \, dx = f(g(x)) + C \)

The key to integration by substitution is locating an expression inside the integral such that \text{the derivative or constant multiple of its derivative is in the integral, also.}\) Doing so allows for the ability to express the differential of the old variable using the differential of the new variable. In general, it works like this:

You are given \( \int f'(g(x))g'(x) \, dx \). By making the substitution \( u = g(x) \), we have

\[
\frac{du}{dx} = g'(x) \implies du = g'(x) \, dx \implies \int f'(g(x))g'(x) \, dx = \int f'(u) \, du = f(u) + C = f(g(x)) + C
\]

Often, instead of \( \int f'(g(x))g'(x) \, dx \), we have \( \int f'(g(x))(k \cdot g'(x)) \, dx \), where \( k \) is a non-zero constant other than one. The constant \( k \) is dealt with when finding \( dx \) in terms of \( du \). Together, regardless of whether \( k \) (non-zero) is equal to one or not, the integral \( \int f'(g(x))(k \cdot g'(x)) \, dx \) can be called a \text{chain-rule integral or composite integral}. Chain rule integrals can be evaluated using integration by substitution.

POLYNOMIAL SUBSTITUTIONS:

The composite integrals \( \int f'(g(x))(k \cdot g'(x)) \, dx \) such that \( g \) is a polynomial can be solved using the substitution \( u = g^-1 \) and then expressing \( dx \) in terms of \( du \).

EX 1) Find the antiderivative: \( \int \frac{2x + 6}{(x^2 + 6x + 11)^5} \, dx \).

SOLUTION: There is an expression in the integral such that its derivative is in the integral, namely \( x^2 + 6x + 11 \).

Therefore, by letting \( u = x^2 + 6x + 11 \), we get

\[
du = (2x + 6) \, dx \implies \int \frac{2x + 6}{(x^2 + 6x + 11)^5} \, dx = \int u^{-5} \, du = \frac{1}{-4}u^{-4} + C = -\frac{1}{4(x^2 + 6x + 11)^4} + C
\]

CHECK:

\[
\frac{d}{dx} \left[ -\frac{1}{4(x^2 + 6x + 11)^4} \right] = \frac{d}{dx} \left[ -\frac{1}{4} \left( x^2 + 6x + 11 \right)^{-4} \right] = -\frac{1}{4} \left( -4 \right) \left( x^2 + 6x + 11 \right)^{-5} \left( x^2 + 6x + 11 \right)' =
\]

\[
\frac{1}{(x^2 + 6x + 11)^5} (2x + 6) = \frac{2x + 6}{(x^2 + 6x + 11)^5}
\]
EX 2) Find the antiderivative: \( \int \frac{x+3}{(x^2+6x+11)^5} \, dx \)

SOLUTION: This time, there is a constant multiple of the derivative of \( x^2+6x+11 \) in the integral.

\[
u = x^2+6x+11 \Rightarrow du = (2x+6) \, dx \Rightarrow \frac{1}{2} \, du = (x+3) \, dx \Rightarrow \int \frac{x+3}{(x^2+6x+11)^5} \, dx = \int \frac{1}{u^5} \cdot \frac{1}{2} \, du = \]

\[
\frac{1}{2} \int u^{-5} \, du = \frac{1}{2} \cdot \frac{1}{-4} u^{-4} + C = -\frac{1}{8(x^2+6x+11)} + C
\]

CHECK:

\[
\frac{d}{dx} \left[ -\frac{1}{8(x^2+6x+11)^4} \right] = \frac{d}{dx} \left[ -\frac{1}{8} (x^2+6x+11)^{-4} \right] = -\frac{1}{8} (-4)(x^2+6x+11)^{-5} (2x+6) = \frac{1}{2} (2x+6)(x^2+6x+11)^{-5} = \frac{x+3}{(x^2+6x+11)^5}
\]

ALTERNATE SOLUTION: Instead of applying a substitution to evaluate chain rule, we can use

\[
\int f'(g(x)) g'(x) \, dx = f(g(x)) + C
\]

By noticing that \( x+3 = \frac{1}{2} (2x+6) \), we can write \( \int \frac{x+3}{(x^2+6x+11)^5} \, dx = \int \frac{1}{2}(2x+6) \, dx \).

Then

\[
\int \frac{1}{2}(2x+6) \, dx = \frac{1}{2} \int 2x+6 \, dx = \frac{1}{2} \int u \, du \quad \left( u = x^2+6x+11 \, du = (2x+6) \, dx \right) =
\]

\[
\frac{1}{2} \cdot \frac{u^4}{-4} + C = -\frac{1}{8u^4} + C = -\frac{1}{8(x^2+6x+11)} + C
\]

We used \( \int f'(g(x)) g'(x) \, dx = f(g(x)) + C \) by multiplying the original integral by an appropriate constant.

EX 3) Find the antiderivative: \( \int \sin 3t \cos^8 3t \, dt \). Note that the integral is almost of the form \( \int u'(u)^8 \, du \), with \( u = \cos 3t \).

This means we will be able to evaluate the integral by using the substitution \( u = \cos 3t \). We have
\[
\begin{align*}
    u &= \cos 3t \Rightarrow du = -3\sin 3tdt \Rightarrow \frac{1}{3}du = \sin 3tdt \Rightarrow \int \sin 3t \cos^3 3tdt = \int u^9 \left( -\frac{1}{3}du \right) = -\frac{1}{3} \int u^9 du = \\
    &\quad \frac{-1}{3} \frac{u^9}{9} + C = \frac{-1}{27} \cos^9 3t + C
\end{align*}
\]

CHECK:
\[
\frac{d}{dt} \left[ -\frac{1}{27} \cos^9 3t \right] = -\frac{1}{27}(9)(\cos^8 3t)(\cos 3t)' = -\frac{1}{3}(\cos^8 3t)(-3\sin 3t) = \sin 3t \cos^8 3t
\]

Alternate Solution: Use \( f' \left( g(x) \right) g'(x) \) after multiplying the integral by an appropriate constant.
\[
\int \sin 3t \cos^8 3tdt = \frac{1}{3} \int 3\sin 3t \cos^8 3tdt = \frac{1}{3} \frac{\cos^9 3t}{9} + C = \frac{1}{27} \cos^9 3t + C
\]

EX 4) Find the antiderivative: \( \int 5xe^{4x^2+3} \) dx.

SOLUTION: Letting \( u = 4x^2 + 3 \), we get
\[
\begin{align*}
    du &= 8xdx \Rightarrow \frac{5}{8} du = 5xdx \Rightarrow \int 5xe^{4x^2+3} dx = \int e^u \frac{5}{8} du = \frac{5}{8} e^u du = \frac{5}{8} e^{4x^2+3} + C
\end{align*}
\]

Check:
\[
\frac{d}{dx} \left[ \frac{5}{8} e^{4x^2+3} \right] = \frac{5}{8} e^{4x^2+3} (4x^2 + 3)' = \frac{5}{8} e^{4x^2+3} (8x) = 5xe^{4x^2+3}
\]

Alternate Solution:
\[
\int 5xe^{4x^2+3} dx = \frac{5}{8} \int 8xe^{4x^2+3} dx = \frac{5}{8} e^{4x^2+3} + C \quad \text{Here } f(x) = e^x \text{ and } g(x) = 4x^2 + 3 \text{ in the statement}
\]
\[
\int f'(g(x))g'(x)dx = f(g(x)) + C
\]

EX 5) Find the antiderivative: \( \int \frac{1+x}{\sqrt{1-x^2}} \) dx.

SOLUTION: There is no substitution apparent here, but splitting the numerator gives us
\[
\int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx. \text{ The first integral can be found by using the basic antiderivative formula}
\]
\[
\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C. \text{ The second integral can be found using the substitution } u = 1-x^2.
\]
\[
\begin{align*}
    u &= 1-x^2 \Rightarrow du = -2xdx \Rightarrow \frac{1}{2} du = xdx \Rightarrow \int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{u}} \left( -\frac{1}{2} du \right) = \frac{-1}{2} \int u^{-1/2} du = \\
    &\quad \frac{-1}{2} \frac{u^{1/2}}{1/2} + C = -\frac{1}{2} \sqrt{u} + c = -\sqrt{1-x^2} + c
\end{align*}
\]
Therefore, \( \int \frac{1 + x}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x - \sqrt{1 - x^2} + C \) [By the way, \( \sqrt{1 - x^2} \) IS NOT \( 1 - x \)!!!!]

EX 6) Find the antiderivative: \( \int \frac{1}{x \ln x} \, dx \)

Solution: Letting \( u = \ln x \), we get

\[
 u = \ln x \Rightarrow \, du = \frac{1}{x} \, dx \Rightarrow \int \frac{1}{x \ln x} \, dx = \int \frac{1}{\ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\ln x| + C
\]

One can also solve the problem by noticing that \( \int \frac{1}{x \ln x} \, dx = \int f'(g(x))g(x) \, dx \), where \( f(x) = \ln x \) and \( g(x) = \ln x \)

EX 7) Find the antiderivative: \( \int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx \)

Solution: Let \( u = e^x - e^{-x} \), then \( du = (e^x + e^{-x}) \, dx \Rightarrow \int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |e^x - e^{-x}| + C \)

EX 8) Find the antiderivative: \( \int \frac{x}{\sqrt[3]{4x + 3}} \, dx \).

Let \( u = 4x + 3 \). Then

\[
x = \frac{1}{4}(u - 3) \Rightarrow \, dx = \frac{1}{4} \, du \Rightarrow \int \frac{x}{\sqrt[3]{4x + 3}} \, dx = \int \frac{1}{4} \frac{x}{u^{1/3}} \, du = \frac{1}{16} \int \frac{u-3}{u^{1/3}} \, du = \frac{1}{16} \int u^{2/3} \, du - \frac{1}{16} \int 3u^{1/3} + C = \frac{1}{16} \frac{1}{5/3} u^{5/3} + C - \frac{3}{16} \frac{1}{2/3} u^{2/3} + C = \frac{3}{80} (4x+3)^{5/3} - \frac{9}{32} (4x+3)^{2/3} + C
\]

EX 9) Find \( \int x^2 (1 + x)^{88} \, dx \)

Solution: Use \( u = 1 + x \) and express the \( x^2 \) factor in terms of \( u \).

\[
x^2 = (u - 1)^2 \text{ and } du = dx \Rightarrow \int x^2 (1 + x)^{88} \, dx = \int (u - 1)^2 u^{88} \, du = \int (u^2 - 2u + 1)u^{88} \, du = \int (u^{90} - 2u^{89} + u^{88}) \, du = \frac{1}{91} u^{91} - \frac{2}{90} u^{90} + \frac{1}{89} u^{89} + C = u^{89} \left( \frac{1}{91} u^2 - \frac{1}{45} u + \frac{1}{89} \right) + C = (1 + x)^{89} \left( \frac{1}{91} (1 + x) - \frac{1}{45} (1 + x) + \frac{1}{89} \right) + C
\]
EX 10) Find $\int \csc(\cos^{-1} x) \, dx$

We can let $y = \cos^{-1} x$ and then use the triangle below to write the integrand without any trigonometric or inverse trigonometric functions.

\[
\int \csc(\cos^{-1} x) \, dx = \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C
\]