Theorem: The set of rational numbers Q is countable.

Proof: We know that the set of integers *Z* is a subset of *Q* and that *Z* is countable. Now we will refer to set *B* as the set of all non-integral rational numbers; that is, $B = Q - Z$. Then $Q = B \cup Z$. If we can show that *B* is countable, then we can conclude that *Q* is countable because a union of two countable sets is countable. We know prove the countability of set B*:*

Let $A_k = \left\{ r \mid r \in Q - Z,\ \ r = \dfrac{p}{q} \text{ and } |p| + q = k \right\}$, where $p \in Z,\ q \in N, \text{ and } k \geq 3,\ k \in N$. Then

$A_3 = \left\{ \pm\frac{1}{2} \right\},\ A_4 = \left\{ \pm\frac{1}{3} \right\},\ A_5 = \left\{ \pm\frac{2}{3}, \pm\frac{3}{2}, \pm\frac{1}{4} \right\},\ A_6 = \left\{ \pm\frac{1}{5} \right\},\ A_7 = \left\{ \pm\frac{1}{6}, \pm\frac{2}{5}, \pm\frac{3}{4}, \pm\frac{5}{2}, \pm\frac{4}{3} \right\}, \ldots$, and so on.

We can express the set *B* as $B = \bigcup_{k=3}^{\infty} A_k$. Now by definition, each A_k is finite and so *B* is a countable set because it is a countable union of countable sets.

$\therefore$ The set of rational numbers is countable.

NOTE: By how each A_k is defined, the numerators of $\dfrac{p}{q}$ can be positive or negative **but** the denominators **are always positive**. So for example, $A_3 = \left\{ \frac{1}{2}, -\frac{1}{2} \right\} = \left\{ \pm\frac{1}{2} \right\} = \left\{ \frac{-1}{2}, \frac{1}{2} \right\}$ because

$p = 1 \text{ or } p = -1 \Rightarrow |p| + q = 1 + 2 = 3$