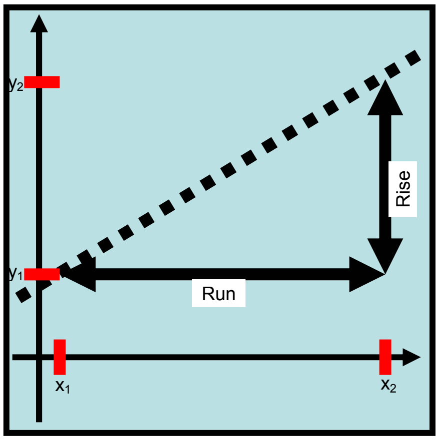
Calculus Review

Slope



Slope is defined as the rise over the run: Slope = rise/run. The rise is the change in the y value and the run is the change in the x value:

Slope = rise/run = $\Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$

Note that the result is the same if the subscripts identifying the points are reversed; for this case, numerator and denominator will both be negative and the negatives cancel. It is critical that the ordered pairs [i.e., (x_1,y_1) and (x_2,y_2)] are kept separate however.

It also does not matter what quadrant the points lie in; the signs work out correctly.

The slope is closely related to the concept of a derivative in Calculus. In the simplest terms, the derivative is the slope at a point on a curve, which is the same as the slope of the tangent to the curve. Standard Leibniz notation for the derivative is based on $\Delta y/\Delta x$; the derivative is written dy/dx.

Exponent Definition

$x^0 = 1$

Derivative of a Line

Recall the equation for a straight line: $y = mx + b$. The slope is m and the y axis intercept is b.

The derivative of a function of the form $y = ax^n + b$ with respect to x is

$dy/dx = a\, n\, x^{(n-1)}$.

(Because b is a constant [think of it as bx^0] its derivative is $0b^{-1} = 0$, and it drops out of the overall derivative.)

For a straight line, a = m and n = 1 so

$dy/dx = m\ 1\ x^{(0)}$, or because $x^0 = 1$,

$dy/dx = m$.

Numerical Derivatives

Numerical approximations of derivatives can be made by taking the slope between points and can serve as a check on the analytical derivative – though usually it will be the other way around. Very often, there is no analytical derivative and all computations must rely on numerical derivatives. There are obvious limitations to this approach. Grid spacing and position of the computed slope need to be considered.

Derivative of a Polynomial

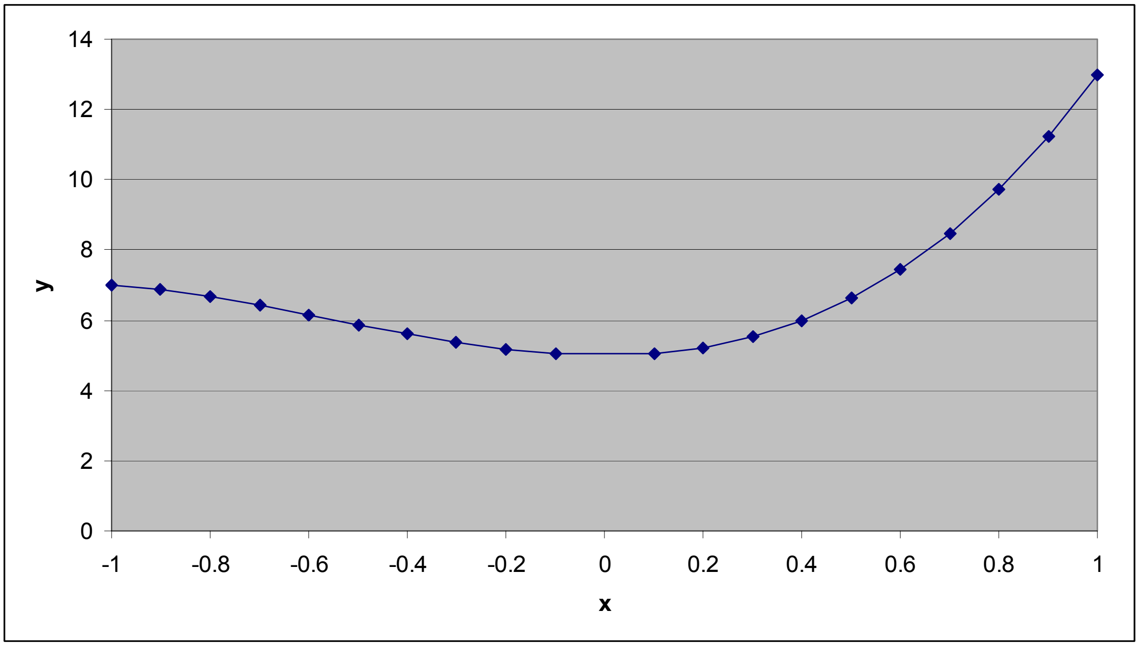
In differential Calculus, we consider the slopes of curves rather than straight lines.

For a general polynomial $y = ax^n + bx^p + cx^q + \ldots$, the derivative with respect to x is

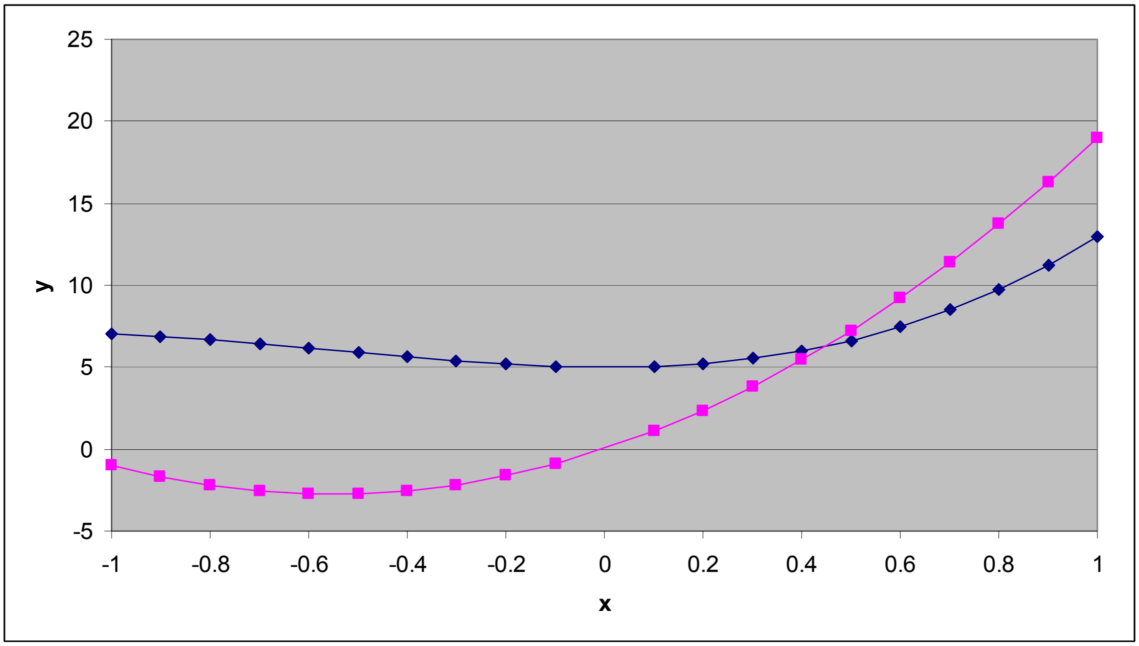
$dy/dx = a\ n\ x^{(n-1)} + b\ p\ x^{(p-1)} + c\ q\ x^{(q-1)} + \ldots$

Here is an example with the following coefficients and exponents:

a	3
n	3
b	5
p	2
c	5
q	0



Note that the slope of this function, instead of being a constant, is itself a function of x. In this case, for x less than 0 the slope is negative and, for x > 0, the slope is positive. The sign and magnitude of the slope are given by the derivative.

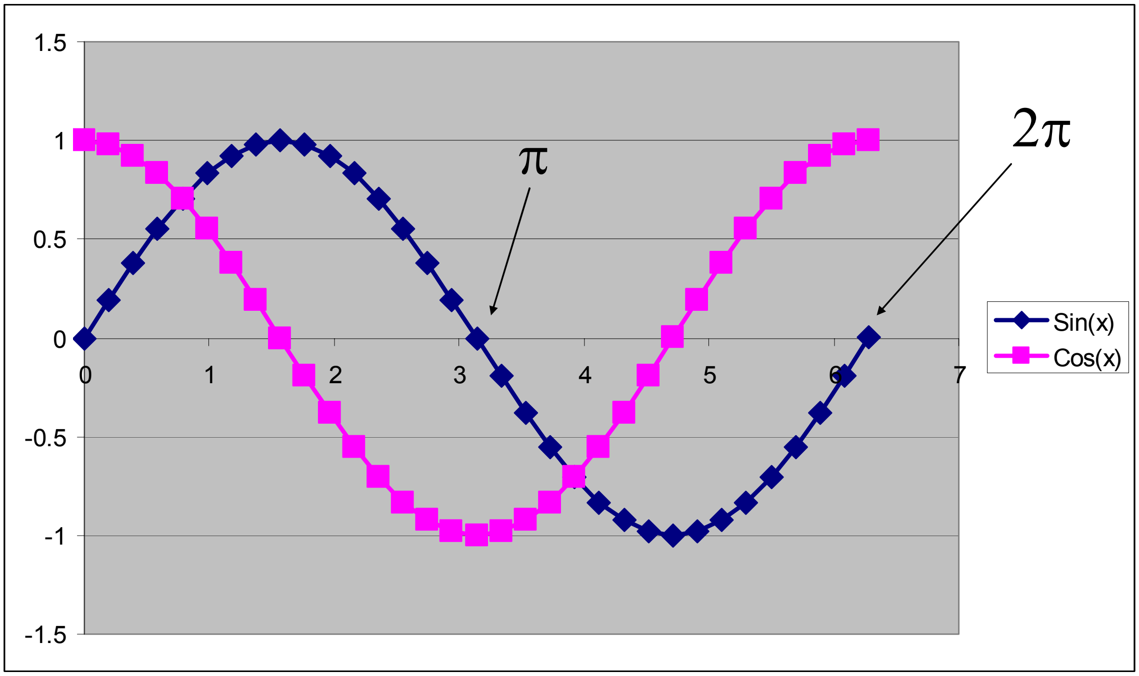


Second derivatives

The second derivative is just that: take the derivative a second time. The notation is d^2h/dx^2.

Derivative of Sine and Cosine

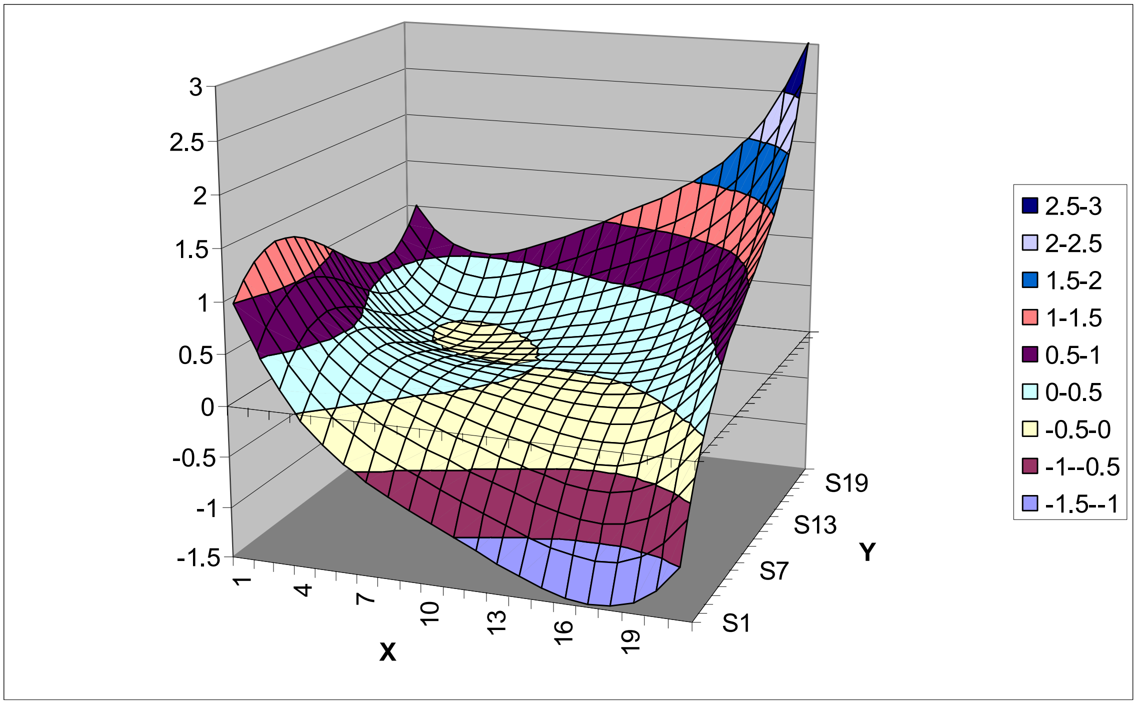
You will also need to know the derivative of the sine and cosine functions for this class. First, always remember that $\sin(0) = 0$ and that the period of both the sine and cosine functions is 2π. Now look at the sine function and notice that its initial slope is positive and that it becomes negative at $\pi/2$. This is exactly the behavior of the cosine and in fact $d(\sin(x))/dx = \cos(x)$. You can also see that the initial slope of the cosine is negative so you can figure out that $d(\cos(x))/dx = -\sin(x)$.



Partial Derivatives

What if the variable of interest – say water table elevation – is a function of both x and y? In this case we use the concept of a partial derivative.

Here is a plot of the function $h(x,y) = x^4 + y^3 + xy$. To compute the partial derivative of h with respect to x at a y location y_0 (indicated by the notation $\delta h/\delta x|_{y=y_0}$), we simply treat any terms containing y only as constants. These constants (like b in the computation of the derivative of the straight line) drop out of the result. If y is in a multiplicative term involving x, it is retained as a constant. Thus $\delta h/\delta x|_{y=y0} = 4x^3 + y_0$.



So what use are these partial derivatives?

One simple and important application is for the computation of gradients. The gradient function of h(x,y) can be written as

del h or **grad** h (bold means vector) which is the same as

$$\nabla h = \mathbf{i}\frac{\partial h}{\partial x} + \mathbf{j}\frac{\partial h}{\partial y}. \qquad (1)$$

i and **j** are the unit vectors in the x and y directions; that is, they point in the positive x and y directions respectively and have length 1. The vector sum of these vectors multiplied by their respective partial derivatives gives the gradient vector. As you will see, partial derivatives play a key role in ground water flow models.

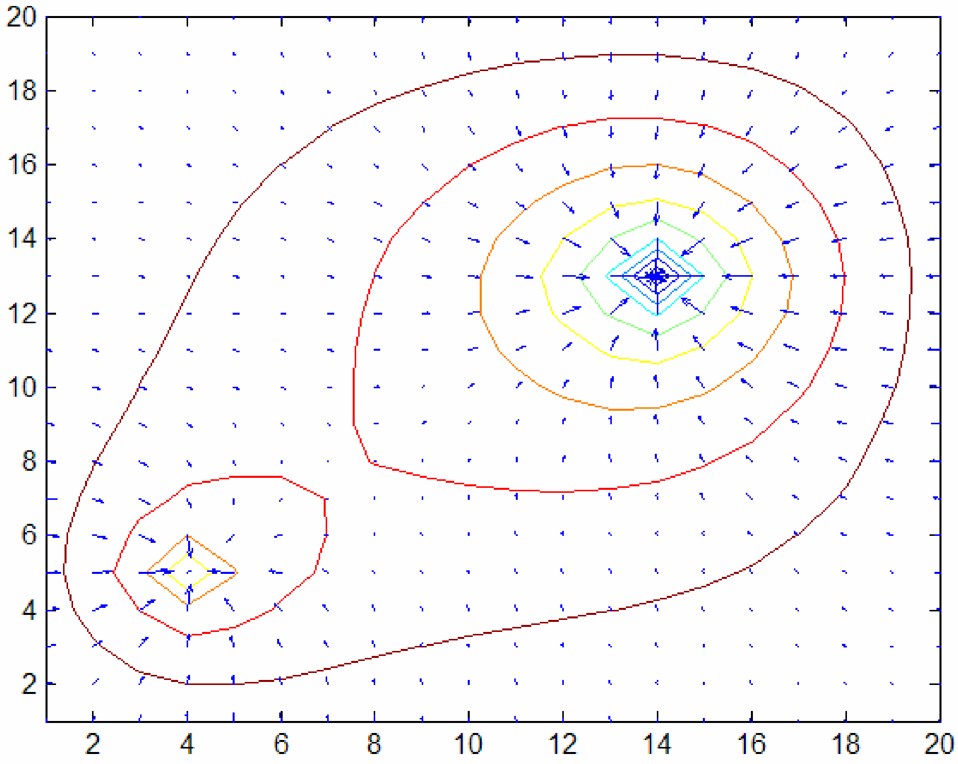
Next, we'll demonstrate the computation of numerical partial derivatives using MATLAB. Let's use a realistic ground water potential surface (i.e., a solution of the Laplace equation that we will derive later).

In MATLAB,

```
h=[];

[dhdx,dhdy]=gradient(h)
contour([1:20],[1:20],h)
hold
```

quiver([1:20],[1:20],-dhdx,-dhdy)



Logarithms

Logarithms (logs for short) are very easy: the (base 10) log of 1000 is 3 (3 zeros), the log of 100 is 2, the log of 10 is 1, and the log of 1 is 0. For numbers less than 1, we have log 0.1 = -1, log 0.01= -2 and log 0.001 = -3.

Logarithms and exponents are very closely related. After all, what is 10 x 10 x 10? It's 1000 and it's also 10^3. We already said that log 1000 = 3. We could write that as log 10^3 = 3 and you can see that it also equals 3 log 10 = 3 x 1 = 3. So in general, log 10^a = a log 10.

Finally, 10^a 10^b = 10^{a+b}.

Assignment

1. You are given the following ordered pairs: Line 1: (1,1), (3,2); Line 2: (1,-1), (3,-2); Line 3: (-1,-1), (-3,-2) ; Line 4: (-1, 1), (-3, 2). Plot the points, compute the slopes.
2. You are given the following ordered pairs: Line 1: (1,1), (-3,-2); Line 2: (1,-1), (-3,2); Line 3: (-1,-1), (1,-1) ; Line 4: (-1, 1), (-1, -1). Plot the points, compute the slopes.

3. Write down the derivative dh/dx of $h = 20x^2 + 10x - 10$. Plot the function and its derivative between $x = -1$ and $x = 1$. Compute and plot a numerical derivative.
4. Write down the derivative dh/dx of $h = x^3 + x^2 + x + 1$. Plot the function and its derivative between $x = -2$ and $x = 2$. Compute and plot a numerical derivative.
5. Write down the second derivative d^2h/dx^2 of $h = x^3 + x^2 + x + 1$. Plot the function and its first and second derivatives between $x = -2$ and $x = 2$.
6. Write down the partial derivative $\delta h/\delta x|_{y=-1}$ of $h(x,y) = x^4 + y^3 + xy$. Plot the function [i.e., h(x,-1)] and the partial derivative between $x = -1$ and 1. Plot the numerical approximation of the derivative.
7. Write down the partial derivative $\delta h/\delta x|_{y=1}$ of $h(x,y) = x^4 + y^3 + xy$. Plot the function [i.e., h(x, 1)] and the partial derivative between $x = -1$ and 1. Plot the numerical approximation of the derivative.
8. Write down the partial derivative $\delta h/\delta x|_{y=-1}$ of $h(x,y) = x^4 + y^3 + x^2y$. Plot the function [i.e., h(x, -1)] and the partial derivative between $x = -1$ and 1. Plot the numerical approximation of the derivative.
9. Write down the partial derivative $\delta h/\delta x|_{y=1}$ of $h(x,y) = x^4 + y^3 + x^2y$. Plot the function [i.e., h(x, 1)] and the partial derivative between $x = -1$ and 1. Plot the numerical approximation of the derivative.
10. Show that $h(x,y) = x^2 - y^2 + xy$ is a solution of the 2-D Laplace Equation [Eqn. (2)]. Taking the first and then the second partial derivatives of h with respect to x and y and then add them. Plot contours and a surface plot of the function h(x,y) using Excel.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (2)$$