MAC 2313 (Calculus III)
Test 1, Thursday September 28, 2006

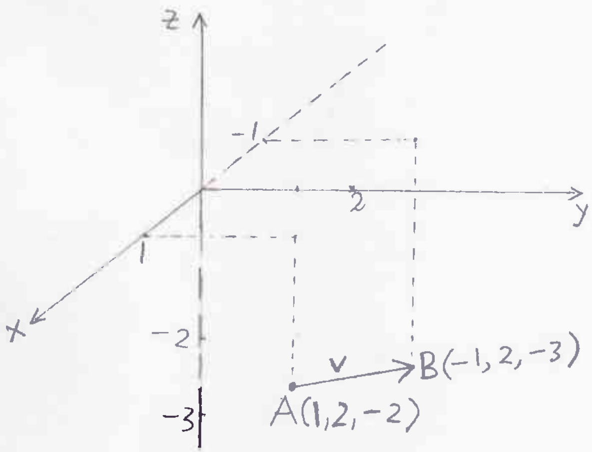
Name: **PID:**

Remember that no documents or graphing calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit by just writing down the answer to any of the problems.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.**

1. [10] Let m be a real number. Consider the surface given by the equation: $x^2+y^2+z^2-2x+2my+4z+9=0$. For which values of m is that surface a sphere? a point?

Eqn of surface may be rewritten $(x-1)^2+(y+m)^2+(z+2)^2=1+m^2+4-9$
or $(x-1)^2+(y+m)^2+(z+2)^2=m^2-4$ (3)
If $m^2>4$, or $m<-2$ or $m>2$; then the surface is the sphere with center $(1,-m,-2)$ and radius $\sqrt{m^2-4}$. (1)
If $m=-2$ or $m=2$ surface reduces to the point $(1,-m,-2)$.
0.5 0.5 1

2. [10] Plot the two points $A(1,2,-2)$, and $B(-1,2,-3)$, and sketch the vector $\mathbf{v}=\overrightarrow{AB}$. What is $||\mathbf{v}||$? Find to the nearest degree the angle that $\mathbf{v}$ makes with the positive x-axis.



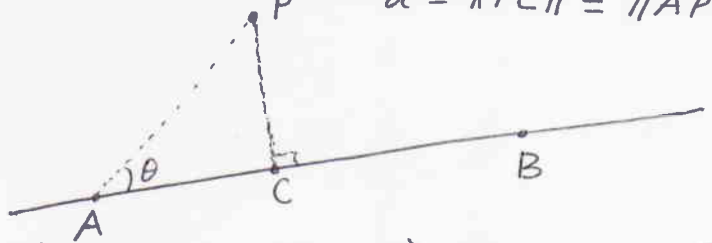
$V=-2\vec{i}-\vec{k}$ 2
$\|V\|=\sqrt{4+1}=\sqrt{5}$ 2
$\cos\theta=\dfrac{V\cdot\vec{i}}{\|V\|\|\vec{i}\|}=\dfrac{-2}{\sqrt{5}}$ 2
$\theta\simeq 153^\circ$ 2

3. [10] If $\mathbf{u}=\vec{i}+2\vec{j}+\vec{k}$, and $\mathbf{b}=-\vec{i}+\vec{j}+3\vec{k}$, find the component of $\mathbf{u}$ that is parallel to $\mathbf{b}$, and the component of $\mathbf{b}$ that is parallel to $\mathbf{u}$.

We may write $u=\alpha b+v$ with v orthogonal to b.
So $u\cdot b=\alpha\|b\|^2+\underbrace{v\cdot b}_{0}$; so $\alpha=\dfrac{u\cdot b}{\|b\|^2}$. The required component of $u \parallel b$ is $\dfrac{u\cdot b}{\|b\|^2}b=\dfrac{-1+2+3}{1+1+9}b=\dfrac{4}{11}(-\vec{i}+\vec{j}+3\vec{k})$
Similarly, the component of $b\parallel u$ is $\dfrac{b\cdot u}{\|u\|^2}u=\dfrac{4}{1+4+1}u=\dfrac{2}{3}(\vec{i}+2\vec{j}+\vec{k})$

4. [10] Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as $d = \frac{||\overrightarrow{AP} \times \overrightarrow{AB}||}{||\overrightarrow{AB}||}$.

$$d = ||\overrightarrow{PC}|| = ||\overrightarrow{AP}|| \sin\theta$$



Now $||\overrightarrow{AP} \times \overrightarrow{AB}|| = ||\overrightarrow{AP}|| \, ||\overrightarrow{AB}|| \sin\theta$; so $\sin\theta = \dfrac{||\overrightarrow{AP} \times \overrightarrow{AB}||}{||\overrightarrow{AP}|| \, ||\overrightarrow{AB}||}$

Hence $d = ||\overrightarrow{AP}|| \cdot \dfrac{||\overrightarrow{AP} \times \overrightarrow{AB}||}{||\overrightarrow{AP}|| \, ||\overrightarrow{AB}||} = \dfrac{||\overrightarrow{AP} \times \overrightarrow{AB}||}{||\overrightarrow{AB}||}$.

5. [10] If θ is the angle between $\mathbf{u} = -2\vec{i} + \vec{j} + \vec{k}$, and $\mathbf{v} = \vec{i} - 2\vec{j} + \vec{k}$, find $\sin\theta$. Is *theta* acute or obtuse?

$||u \times v|| = ||u|| \, ||v|| \sin\theta$; so $\sin\theta = \dfrac{||u \times v||}{||u|| \, ||v||}$ 1

Now $u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 3\vec{i} + 3\vec{j} + 3\vec{k}$ 3

So

$$\sin\theta = \frac{\sqrt{9+9+9}}{\sqrt{4+1+1} \cdot \sqrt{1+4+1}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}.$$ 1

The sign of $u \cdot v$ will tell us whether θ is acute or obtuse.

Now $u \cdot v = -2 - 2 + 1 = -3 < 0$; so θ is obtuse.

2 1 2

6. [10] Show that the two lines $L_1 : x = 1 - t, \quad y = -1, \quad z = t$, and $L_2 : x = 1, \quad y = t, \quad z = 2 + 2t$ intersect, and find their point of intersection.

Do there exist t_1, t_2 such that

$1 - t_1 = 1 \to t_1 = 0$

$-1 = t_2 \to t_2 = -1$ } reporting these values in last eqn yields

$t_1 = 2 + 2t_2$? $\quad 0 = 2 + 2(-1)$ which is true

So L_1 and L_2 intersect at $(1, -1, 0)$.

7. [12] a) Convert $(3/2, \sqrt{3}/2, -1)$ from rectangular coordinates to spherical coordinates. b) Convert the equation $x^2 - y^2 - z^2 = 1$ from rectangular coordinates to cylindrical coordinates.

a) $\rho = \sqrt{(\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2 + (-1)^2} = \sqrt{\frac{9}{4} + \frac{3}{4} + 1} = \sqrt{4} = 2$ 2

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{3/2}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ 2

$\varphi = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ 2

b) $x = r\cos\theta \quad y = r\sin\theta, \; z = z$

So eqn becomes $r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 1$, or $r^2(\cos^2\theta - \sin^2\theta) - z^2 = 1$ 4

or $\underline{\underline{r^2\cos(2\theta) = z^2 + 1}}$

2

8. [20] If $r(t) = e^{-t}\vec{i} + \sin(1 - e^{-t})\vec{j} + \cos(1 - e^{-t})\vec{k}$, find **T**, **N**, **B**, as well as the equations of the osculating, and the rectifying planes at time $t = 0$.

$r'(t) = -e^{-t}\vec{i} + e^{-t}\cos(1 - e^{-t})\vec{j} - e^{-t}\sin(1 - e^{-t})\vec{k}$

$\|r'(t)\| = e^{-t}\sqrt{1 + \cos^2(1 - e^{-t}) + \sin^2(1 - e^{-t})} = \sqrt{2}\,e^{-t}$

$T(0) = r'(0)/\|r'(0)\| = \frac{-\vec{i} + \vec{j}}{\sqrt{2}}$. $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{-\vec{i} + \cos(1 - e^{-t})\vec{j} - \sin(1 - e^{-t})\vec{k}}{\sqrt{2}}$

$T'(t) = \frac{-e^{-t}\sin(1 - e^{-t})\vec{j} - e^{-t}\cos(1 - e^{-t})\vec{k}}{\sqrt{2}}$

$N(0) = \frac{T'(0)}{\|T'(0)\|} = \frac{-\vec{k}}{\sqrt{2}} \Big/ \frac{1}{\sqrt{2}} = -\vec{k}$, $B(0) = T(0) \times N(0) = \frac{(-\vec{i} + \vec{j}) \times (-\vec{k})}{\sqrt{2}}$

$= \frac{\vec{i} \times \vec{k} - \vec{j} \times \vec{k}}{\sqrt{2}} = \frac{-\vec{j} - \vec{i}}{\sqrt{2}}$

Planes passes through $r(0) = (1, 0, 1)$

Eqn of osc. plane: $-(x - 1) - y = 0$ or $x + y - 1 = 0$

Eqn of rect. plane: $-(z - 1) = 0$ or $\underline{\underline{z = 1}}$

9. [10] If $r(t) = \sqrt{3}e^t\vec{i} + \sin(e^t)\vec{j} + \cos(e^t)\vec{k}$, find an arc length parametrization of the curve that has the same orientation as the given curve, and has $t = 0$ as the reference point.

arc length parameter is $s = \int_0^t \|r'(u)\|\,du = \int_0^t \sqrt{3e^{2u} + e^{2u}\cos^2(e^u) + e^{2u}\sin^2(e^u)}\,du$

$= 2\int_0^t e^u\,du$

$= 2e^u\Big]_0^t = 2(e^t - 1)$

$e^t = \frac{s+2}{2}$

$r(s) = \frac{\sqrt{3}}{2}(s+2)\vec{i} + \sin\left(\frac{s+2}{2}\right)\vec{j} + \cos\left(\frac{s+2}{2}\right)\vec{k}$