Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

where x is a variable and $\{a_n\}$ is a sequence of real numbers. a_n are called coefficients of the power series

For each x, the series $\sum_{n=0}^{\infty} a_n x^n$ converges or diverges. We can define a function f(x) as the sum of the series, that is $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ and the domain of f is the set of all x for which the series converges.

Example

Series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ is geometric with r = x, and therefore convergent for | x | < 1. Its sum is $\frac{1}{1-x}$, when |x| < 1, therefore we can write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{if } -1 < x < 1$$

More generally, a series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$ is called a **power series in (x-c)** or a power series centered at c

Note that when x = c (x - c = 0), the series converges (to a_0). Thus, *every power series centered at c converges at c.*

We use the *Ratio or Root Test for Absolute Convergence* to find where a power series converges.

Ex Find all values of x for which the series converges

a) $\sum_{n=0}^{\infty} n! x^n$

b) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Theorem

For a given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ there are only three possibilities

(I) The series converges only when x = c

(II) The series converges for all x

(III) There is a positive number R (called the **radius of convergence**), such that the series converges when | x – c| < R and diverges when |x-c| > R

Remark: in case (I), the radius of convergence is R = 0
In case (II), the radius of convergence is R = ∞

The interval I that contains all values of x for which a power series converges is called the **interval of convergence.**

In case (I), the interval of convergence is just a point {c}, I = {c}
In case (II), the interval of convergence is (-∞, +∞), I = (-∞, +∞)
In case (III), the interval of convergence has the endpoints c – R and c + R, the endpoints can be included or excluded

Theorem

Let $\sum_{n=0}^{\infty} a_n (x-c)^n$ be a power series and let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ (or $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$). Then the radius of convergence R is $R = \frac{1}{L}$. The series converges for | x-c| < R and diverges for |x-c|> R. The convergence of the series when | x – c| = R must be examined separately

(we will use here the convention $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$)

Ex Find the radius of convergence and the interval of convergence of the following series

a) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

b) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n}$

Representation of functions by power series

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$ for all x in the interval of convergence.

We say that f(x) is represented by the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$

Example Since

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$, -1 < x < 1

We can say that the function $\frac{1}{1-x}$ is represented by the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$, when |x| < 1

Ex. Use the power series representation of $\frac{1}{1-x}$, to find power series representation of

a) $\frac{1}{1+x}$

b) $\dfrac{1}{1+x^2}$

c) $\dfrac{1}{2+x}$

d) $\dfrac{x^3}{2+x}$

Theorem

Suppose $\sum_{n=0}^{\infty} a_n (x-c)^n$ has the radius of convergence R > 0 and let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for |x – c | < R.

Then

(I) f is a differentiable (and therefore continuous) function on (c-R, c+R) and

$$\frac{df(x)}{dx} = \sum_{n=0}^{\infty} \frac{d\left(a_n (x-c)^n\right)}{dx} = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

And

(II) f is integrable on any subinterval of (c-R, c+R) and

$$\int f(x)dx = \sum_{n=0}^{\infty} \int a_n (x-c)^n dx = \sum_{n=1}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

All the series mentioned above are convergent when |x – c| < R

Remark: This theorem simply states that if f has a power series representation , then to find the derivative (or integral) we differentiate (integrate) the series term by term.

Ex. Find the power series representation for

a) ln(1 +x)

b) arctanx

c) $\dfrac{1}{(1-x)^2}$