Homework 12:  Chapter 4:  4.17; 4.20; 4.24; 4.40

Problem 4.17

(a) $V(r) = -\frac{GMm}{r}$. So $\frac{e^2}{4\pi\varepsilon_0} \rightarrow GMm$ translates hydrogen results to the gravitational analogs.

(b) Equation 4.72: $a = \left(\frac{4\pi\varepsilon_0}{e^2}\right) \frac{h^2}{m}$, so $a_g = \frac{h^2}{GMm^2}$

$$a_g = \frac{\left(1.0546 \times 10^{-34}\right)^2}{(6.67726 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(1.9892 \times 10^{10} \text{ kg})(5.98 \times 10^{24} \text{ kg})^2} = 2.34 \times 10^{-138} \text{ m}.$$  

(c) Equation 4.70: $E_n = -\left[\frac{m}{2h^2}(GMm)^2\right] \frac{1}{r_n^2}$. But $\frac{GMm}{r_o^2} = \frac{mv^2}{r_o} \rightarrow \frac{1}{2}mv^2 = \frac{GMm}{2r_o}$, so

$E_o = \frac{GMm}{2r_o} = -\left[\frac{m}{2h^2}(GMm)^2\right] \frac{1}{n^2}$ \Rightarrow $n^2 = \frac{GMm^2}{h^2} r_o \rightarrow n = \sqrt{\frac{r_o}{a_g}}.$

$r_o = \text{earth-sun distance} = 1.496 \times 10^{11} \text{ m} \Rightarrow n = \sqrt{\frac{1.496 \times 10^{11}}{2.34 \times 10^{-138}}} = 2.53 \times 10^{74}.$

(d)

$$\Delta E = -\left[\frac{G^2M^2m^3}{2h^2}\right] \left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right].$$

So $\frac{1}{n+1)^2} \sim \frac{1}{n^2} \left[1 - \frac{2}{n} \right] = -\frac{2}{n^2}; \quad \Delta E = \frac{G^2M^2m^2}{h^2} \frac{1}{n^2}.$

$$\Delta E = \frac{(6.67 \times 10^{-11})^2(1.99 \times 10^{30})^2(5.98 \times 10^{24})}{(1.055 \times 10^{-34})(2.53 \times 74)^3} = 2.00 \times 10^{-41} \text{ J}. \quad E_o = \Delta E = h\nu = \frac{hc}{\lambda}.$$  

$$\lambda = (3 \times 10^8)(6.63 \times 10^{-34})/(2.00 \times 10^{-41}) = 9.52 \times 10^{15} \text{ m}.$$  

But 1 ly = 9.46 \times 10^{15} m. Is it a coincidence that $\lambda \approx 1$ ly? No: From part (c), $n^2 = GMm^2r_o/h^2$, so

$$\frac{\lambda}{\Delta E} = \frac{c\hbar}{G^2M^2m^3} = \frac{2\pi\hbar^2}{G^2M^2m^3} \left(\frac{GMm^2r_o}{h^2}\right)^{3/2} = c\left(\frac{\sqrt{r_o^3}}{GM}\right).$$  

But (from (c)) $v = \sqrt{GM/r_o} = 2\pi r_o/T$, where $T$ is the period of the orbit (in this case one year), so $T = 2\pi\sqrt{r_o^3}/GM$, and hence $\lambda = cT$ (one light year). [Incidentally, the same goes for hydrogen: The wavelength of the photon emitted in a transition from a highly excited state to the next lower one is equal to the distance light would travel in one orbital period.]
Problem 4.20

(a)  
Equation 3.71 ⇒ \( \frac{d(L_x)}{dt} = \frac{i}{\hbar} \{ [H, L_x] \} \).  
\[ [H, L_x] = [V, y p_z - z p_y] = y [V, p_z] - z [V, p_y] \]  
The first term is zero (Problem 4.19(c)); the second would be too if \( V \) were a function only of \( r = |r| \), but in general  
\[ [H, L_x] = [V, y p_z - z p_y] = y [V, p_z] - z [V, p_y]. \]  
Now (Problem 3.13(c)):
\[ [V, p_z] = i \hbar \frac{\partial V}{\partial z} \text{ and } [V, p_y] = i \hbar \frac{\partial V}{\partial y}. \]  
So \( [H, L_x] = y i \hbar \frac{\partial V}{\partial z} - z i \hbar \frac{\partial V}{\partial y} = i \hbar r \times (\nabla V)_z \).

Thus \( \frac{d(L_x)}{dt} = -\langle [r \times (\nabla V)]_z \rangle \), and the same goes for the other two components:
\[ \frac{d(L_x)}{dt} = \langle [r \times (-\nabla V)] \rangle = \langle N \rangle. \]  
QED

(b)  
If \( V(\mathbf{r}) = V(r) \), then \( \nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}}, \) and \( r \times \hat{\mathbf{r}} = 0 \), so \( \frac{d(L_x)}{dt} = 0 \).  
QED

Problem 4.24

(a)  
\( H = 2 \left( \frac{1}{2} m \nu^2 \right) = m \nu^2; \quad |L| = \sqrt{\frac{a^2}{2} m \nu^2} = a m \nu, \quad \text{so } L^2 = a^2 m^2 \nu^2, \quad \text{and hence } H = \frac{L^2}{m \nu^2}. \)

But we know the eigenvalues of \( L^2 : h^2 (l + 1); \) or, since we usually label energies with \( n \):
\[ E_n = \frac{\hbar^2 n (n + 1)}{m \nu^2} \quad (n = 0, 1, 2, \ldots). \]

(b)  
\( \psi_{nm}(\theta, \phi) = Y_n^m(\theta, \phi) \) the ordinary spherical harmonics. The degeneracy of the \( n \)th energy level is the number of \( m \)-values for given \( n \): \( 2n + 1. \)
Problem 4.40

(a)
\[
\frac{d}{dt} (r \cdot p) = \frac{i}{\hbar} [H, r \cdot p].
\]

\[
[H, r \cdot p] = \sum_{i=1}^{3} [H, r_i p_i] = \sum_{i=1}^{3} \left( [H, r_i] p_i + r_i [H, p_i] \right) = \sum_{i=1}^{3} \left( \frac{1}{2m} [p_i^2, r_i] p_i + r_i [V, p_i] \right),
\]

\[
[p_i^2, r_i] = \sum_{j=1}^{3} [p_j p_j, r_i] = \sum_{j=1}^{3} (p_j [p_j, r_i] + [p_j, r_i] p_j) = \sum_{j=1}^{3} [p_j (-i\delta_{ij}) + (-i\delta_{ij}) p_j] = -2i\hbar p_i.
\]

\[
[V, p_i] = i\hbar \frac{\partial V}{\partial r_i} \quad \text{(Problem 3.13(c)).}
\]

\[
[H, r \cdot p] = \sum_{i=1}^{3} \left[ \frac{1}{2m} (-2i\hbar) p_i p_i + r_i \left( i\hbar \frac{\partial V}{\partial r_i} \right) \right]
\]

\[
= i\hbar \left( -\frac{p^2}{m} + r \cdot \nabla V \right). \quad \frac{d}{dt} (r \cdot p) = \left( \frac{p^2}{m} - r \cdot \nabla V \right) = 2\langle T \rangle - \langle r \cdot \nabla V \rangle.
\]

For stationary states \(\frac{d}{dt} (r \cdot p) = 0\), so \(2\langle T \rangle = \langle r \cdot \nabla V \rangle\). \text{QED}

(b)

\[
V(r) = -\frac{\epsilon^2}{4\pi\epsilon_0 r} \Rightarrow \nabla V = \frac{\epsilon^2}{4\pi\epsilon_0} \frac{1}{r^2} \Rightarrow r \cdot \nabla V = \frac{\epsilon^2}{4\pi\epsilon_0} \frac{1}{r} = -V. \quad \text{So } 2\langle T \rangle = -\langle V \rangle.
\]

But \(\langle T \rangle = \langle V \rangle = E_n\), so \(2\langle T \rangle = E_n\), or \(\langle T \rangle = -E_n\); \(\langle V \rangle = 2E_n\). \text{QED}

(c)

\[
V = \frac{1}{2} m \omega^2 r^2 \Rightarrow \nabla V = m \omega^2 r \Rightarrow r \cdot \nabla V = m \omega^2 r^2 = 2V. \quad \text{So } 2\langle T \rangle = 2\langle V \rangle, \text{ or } \langle T \rangle = \langle V \rangle.
\]

But \(\langle T \rangle + \langle V \rangle = E_n\), so \(\langle T \rangle = \langle V \rangle = \frac{1}{2} E_n\). \text{QED}