1. Total energy density

(a)
$$\rho(T)=\frac{8\pi h}{c^3}\int_0^\infty d\nu\,\frac{\nu^3}{e^{\frac{h\nu}{kT}}-1}=\frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty dx\,\frac{x^3}{e^x-1}$$

$$\int_0^\infty dx\,\frac{x^3}{e^x-1}=\int_0^\infty x^3e^{-x}dx\sum_{n=0}^\infty e^{-nx}=\sum_{n=0}^\infty\frac{1}{(n+1)^4}\int_0^\infty dy\,y^3e^{-y}$$

$$=6\sum_{n=1}^\infty\frac{1}{n^4}=\frac{\pi^4}{15}$$

$$\rho(T)=\frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\cdot\frac{\pi^4}{15}=\frac{8\pi^5k^4}{15h^3c^3}T^4=\sigma T^4$$

with $\sigma\equiv\dfrac{8\pi^5k^4}{15h^3c^3}=\text{const.}=7.5662\times10^{-16}\,J/m^3K$

(b) Planck formulas $\rho(\nu,T)=\dfrac{8\pi h}{c^3}\dfrac{\nu^3}{e^{h\nu/kT}-1}$

Rayleigh-Jeans formula $\rho(\nu,T)=\dfrac{8\pi kT}{c^3}\nu^2$

↑ from continuous radiation

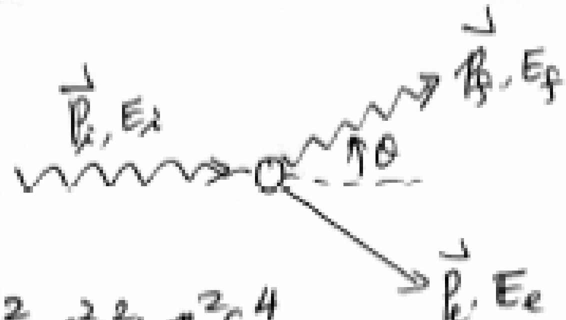
For $x=\frac{h\nu}{kT}\ll1$ $\quad e^x\simeq1+x$

$$e^{\frac{h\nu}{kT}}-1\simeq\frac{h\nu}{kT}$$

$$\rho(\nu,T)=\frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}\simeq\frac{8\pi h}{c^3}\frac{\nu^3}{\frac{h\nu}{kT}}=\frac{8\pi kT}{c^3}\nu^2$$

When $h\nu\ll kT$, quantization diminishes!

2.



$$E^2 = p^2c^2 + m_0^2c^4$$

For photon. $E = pc$ $\qquad \vec{P}_{e,\text{initial}} = 0$.

momentum conservation $\quad \vec{P}_i = \vec{P}_f + \vec{P}_e \quad$ ①

energy conservation $\quad E_i + m_ec^2 = E_f + E_e \quad$ ②

From eq.①. $\vec{P}_e = \vec{P}_i - \vec{P}_f$

$$P_e^2 = P_i^2 + P_f^2 - 2P_iP_f\cos\theta \quad ③$$

eq②. $E_e^2 = (E_i - E_f + m_ec^2)^2$ $\downarrow$

use $\begin{cases} E_i = P_ic \\ E_f = P_fc \\ E_e^2 = P_e^2c^2 + m_e^2c^4 \end{cases}$

$$P_e^2 = \underbrace{(P_i - P_f)^2} + 2(P_i - P_f)m_ec$$

$\downarrow$ ③

$$\cancel{P_i^2} + \cancel{P_f^2} - 2P_iP_f\cos\theta = \cancel{P_i^2} + \cancel{P_f^2} - 2P_iP_f + 2(P_i - P_f)m_ec$$

$$2P_iP_f(1-\cos\theta) = 2(P_i - P_f)m_ec.$$

$$P_i - P_f = \frac{P_iP_f}{m_ec}(1-\cos\theta)$$

$$\frac{1}{P_f} - \frac{1}{P_i} = \frac{1}{m_ec}(1-\cos\theta) \xRightarrow[P=\frac{h}{\lambda}]{} \boxed{\lambda_f - \lambda_i = \frac{h}{m_ec}(1-\cos\theta)}$$

3. $E = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2 \equiv \frac{1}{2} m\omega^2 a^2 \quad (x = a\sin\omega t)$

$$p^2 = 2m\left(E - \frac{1}{2} m\omega^2 x^2\right) = m^2\omega^2 (a^2 - x^2)$$

$$p = m\omega\sqrt{a^2 - x^2}$$

$$\oint p\,dx = 2\int_{-a}^{a} p\,dx = 2m\omega \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= 2m\omega \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_{-a}^{a}$$

$$= \pi m\omega a^2 = n h$$

$$E = \frac{1}{2} m\omega^2 a^2 = n\frac{h}{2\pi}\omega = n\hbar\omega.$$

OR

$$\oint p\,dx = \oint m\dot{x}\,dx = \oint m\,(a\omega\cos\omega t)\,d(a\sin\omega t)$$

$$= m\omega^2 a^2 \oint \cos^2\omega t\,dt$$

$$= m\omega a^2 \oint \cos^2 y\,dy$$

$$= m\omega a^2 \pi = n h$$

$$E = \frac{1}{2} m\omega^2 a^2 = n\hbar\omega.$$

4. Consider oscillations give rise to radiation, and assume:

$n(\nu) \equiv$ the number density of harmonic oscillators

$\langle E \rangle \equiv$ averge energy of each oscillator

$\rho(\nu, T) \equiv$ energy density of radiation

$$\rho(\nu,T) = n(\nu) \cdot \langle E \rangle$$

From simple argurement based on standing waves as the radiation in equilibrium: $n(\nu) \equiv \dfrac{8\pi\nu^2}{c^3}$

Postulate energy radiation of oscillators comes only in integer multiples of $h\nu$.

$$E_n = nh\nu \quad (n = 0, 1, 2 \cdots)$$

$$\langle E \rangle = \sum_n nh\nu e^{-nh\nu/kT} \Big/ \sum_n e^{-nh\nu/kT} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

Finally:

$$\rho(\nu,T) = n(\nu) \cdot \langle E \rangle$$

$$= \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1}$$