Homework 2.

1. $$\int_{-\infty}^{\infty}|\phi(p,t)|^2 dp = \int_{-\infty}^{\infty}\phi^*(p,t)\,\phi(p,t)\,dp$$

$$= \int_{-\infty}^{\infty}dp\,\phi^*(p,t)\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dx\,\psi(x,t)\,e^{-ipx/\hbar}$$

$$= \int_{-\infty}^{\infty}dx\,\psi(x,t)\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dp\,\phi^*(p,t)\,e^{-ipx/\hbar}$$

$$= \int_{-\infty}^{\infty}dx\,\psi(x,t)\,\psi^*(x,t) = \int_{-\infty}^{\infty}|\psi(x,t)|^2 = 1$$

2. $$\langle p\rangle - \langle p\rangle^* = \int_{-\infty}^{\infty}dx\left[\psi^*\frac{\hbar}{i}\frac{\partial\psi}{\partial x} - \psi\left(-\frac{\hbar}{i}\right)\frac{\partial\psi^*}{\partial x}\right]$$

$$= \frac{\hbar}{i}\int_{-\infty}^{\infty}dx\left\{\psi^*\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi^*}{\partial x}\right\}$$

$$= \frac{\hbar}{i}\int_{-\infty}^{\infty}dx\,\frac{\partial}{\partial x}(\psi^*\psi) = \frac{\hbar}{i}\Big[\psi^*\psi\Big]_{-\infty}^{\infty} = 0$$

$$\therefore \langle p\rangle = \langle p\rangle^*$$

3. (a) $$1 = \int_{-\infty}^{\infty}A\,e^{-\lambda(x-a)^2}dx \quad \text{let } u \equiv x-a \quad du = dx$$

$$= A\int_{-\infty}^{\infty}e^{-\lambda u^2}du = A\sqrt{\frac{\pi}{\lambda}} \qquad A = \sqrt{\frac{\lambda}{\pi}}$$

(b) $$\langle x\rangle = A\int_{-\infty}^{\infty}x\,e^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du$$

$$= A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = a$$

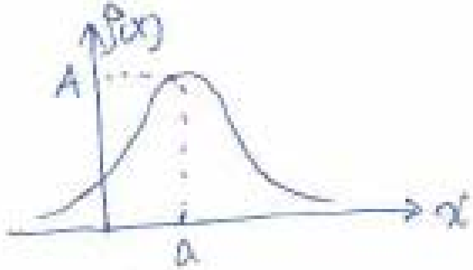
$$\langle x^2\rangle = A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}dx$$

$$= A\left\{\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} u e^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\}$$

$$= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = a^2 + \frac{1}{2\lambda}$$

$$\langle \Delta x\rangle^2 = \sigma^2 \equiv \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda} \qquad \Delta x = \frac{1}{\sqrt{2\lambda}}$$

(c)



4. $$1 = \int_{-\infty}^{\infty} |\phi(k)|^2 dk = |A|^2\int_{-\infty}^{\infty} e^{-\frac{a^2}{2}(k-k_0)^2}dk \qquad y \equiv k - k_0$$

$$= |A|^2\int_{-\infty}^{\infty} e^{-a^2y^2/2}dy = \frac{\sqrt{2\pi}}{a} \qquad A = \left(\frac{a^2}{2\pi}\right)^{\frac{1}{4}}$$

$$\phi(k) = \left(\frac{a^2}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{a^2}{4}(k-k_0)^2}$$

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}dk = \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\frac{a^2}{4}(k-k_0)^2 + ikx}dk$$

$$\because -\frac{a^2}{4}(k-k_0)^2 + ikx = -\left[\frac{a}{2}(k-k_0) - \frac{ix}{a}\right]^2 - \frac{x^2}{a^2} + ik_0x$$

$$y = \frac{a}{2}(k-k_0) - \frac{ix}{a} \qquad dk = \frac{2}{a}dy$$

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}} e^{ik_0x} e^{-y^2}\left(\frac{2}{a}\right)dy$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{a^2}} e^{ik_0x}\int_{-\infty}^{\infty} e^{-y^2}dy = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} e^{-\frac{x^2}{a^2}} e^{ik_0x}$$

Probability $P = \int_{-\frac{a}{2}}^{\frac{a}{2}} |\psi_0(x)|^2 dx = \sqrt{\frac{2}{\pi a^2}}\int_{-\infty}^{\infty} e^{-\frac{2x^2}{a^2}} dx = \frac{2}{3}$

(b) $$1 = \int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = |A|^2\int_{-a}^{a} e^{-ik_0x} e^{ik_0x} dx = |A|^2\int_{-a}^{a} dx = 2a|A|^2$$

$$A = \frac{1}{\sqrt{2a}}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi_0(x) e^{-ikx} dx = \frac{1}{2\sqrt{\pi a}}\int_{-a}^{a} e^{ik_0x} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{\pi a}}\,\frac{\sin[(k-k_0)a]}{(k-k_0)}$$