Problem 2.10

(a) Using Eqs. 2.47 and 2.59,

$$a_+\psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x + m\omega x\right] e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}.$$

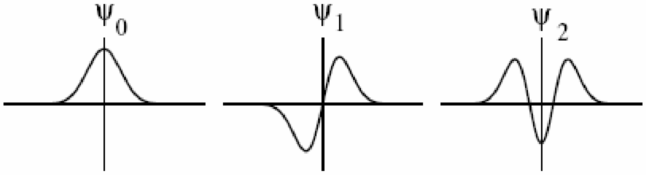
$$(a_+)^2\psi_0 = \frac{1}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega\left(-\hbar\frac{d}{dx} + m\omega x\right) x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(1 - x\frac{m\omega}{2\hbar}2x\right) + m\omega x^2\right] e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) e^{-\frac{m\omega}{2\hbar}x^2}.$$

Therefore, from Eq. 2.67,

$$\psi_2 = \frac{1}{\sqrt{2}}(a_+)^2\psi_0 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) e^{-\frac{m\omega}{2\hbar}x^2}.}$$

(b)



(c) Since ψ_0 and ψ_2 are even, whereas ψ_1 is odd, $\int \psi_0^*\psi_1 dx$ and $\int \psi_2^*\psi_1 dx$ vanish automatically. The only one we need to check is $\int \psi_2^*\psi_0\, dx$:

$$\int \psi_2^*\psi_0\, dx = \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx - \frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx\right)$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right) = 0.\ \checkmark$$

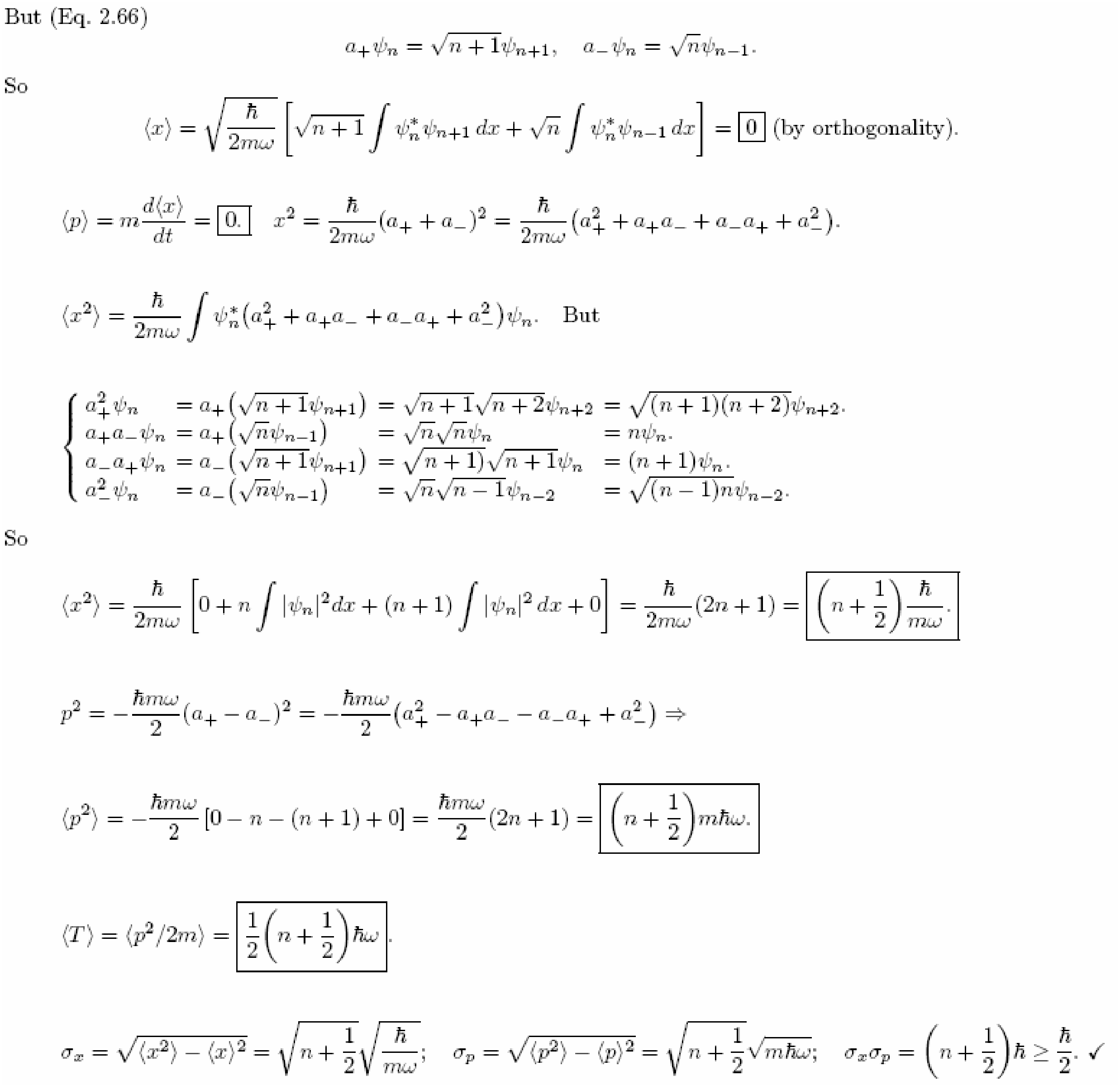
Problem 2.12

From Eq. 2.69,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-),$$

so

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int \psi_n^*(a_+ + a_-)\psi_n\, dx.$$



But (Eq. 2.66)

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\,\psi_{n-1}.$$

So

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = \boxed{0}\ \text{(by orthogonality)}.$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right).$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int \psi_n^*\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)\psi_n. \quad \text{But}$$

$$\begin{cases} a_+^2\psi_n & = a_+\left(\sqrt{n+1}\,\psi_{n+1}\right) & = \sqrt{n+1}\sqrt{n+2}\,\psi_{n+2} & = \sqrt{(n+1)(n+2)}\,\psi_{n+2}. \\ a_+a_-\psi_n & = a_+\left(\sqrt{n}\,\psi_{n-1}\right) & = \sqrt{n}\sqrt{n}\,\psi_n & = n\psi_n. \\ a_-a_+\psi_n & = a_-\left(\sqrt{n+1}\,\psi_{n+1}\right) & = \sqrt{n+1)}\sqrt{n+1}\,\psi_n & = (n+1)\psi_n. \\ a_-^2\psi_n & = a_-\left(\sqrt{n}\,\psi_{n-1}\right) & = \sqrt{n}\sqrt{n-1}\,\psi_{n-2} & = \sqrt{(n-1)n}\,\psi_{n-2}. \end{cases}$$

So

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[0 + n\int|\psi_n|^2dx + (n+1)\int|\psi_n|^2\,dx + 0\right] = \frac{\hbar}{2m\omega}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}.}$$

$$p^2 = -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right) \Rightarrow$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\left[0 - n - (n+1) + 0\right] = \frac{\hbar m\omega}{2}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)m\hbar\omega.}$$

$$\langle T\rangle = \langle p^2/2m\rangle = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}.$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\frac{\hbar}{m\omega}}; \quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{m\hbar\omega}; \quad \sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}. \ \checkmark$$

3. One major difference between classical and quantum mechanics on a harmonic oscillator is that in quantum mechanics you may find probability of the particle until $|x| \to \infty$, while in classical mechanics the particle can only be in so-called "classical range $-x_c < x < x_c$" which is defined as: $\frac{1}{2}m\omega^2 x_c^2 \leq E$. For the ground state, find the total probability of the particle beyond the "classical range".

SOLUTION:

Find out what x_c is. Then the probability is

$$w = 2\int_{x_c}^{\infty}|\psi_0|^2\,dx = 2(1-\int_0^{x_c}|\psi_0|^2\,dx)$$

then you have to use error function to get the final value of w ~ 0.16.

4. The Hamiltonian of a coupled harmonic oscillator can be expressed as $H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + \lambda x_1 x_2$ where $p_1 = -i\hbar\frac{\partial}{\partial x_1}$ and $p_2 = -i\hbar\frac{\partial}{\partial x_2}$.
   Assume $\lambda < m\omega^2$, find the energy levels of the coupled oscillator.

SOLUTION:

For such linear coupled oscillators, you need to do a variable transformation on both x_1 and x_2, such that in the new set of variables the “quasi” oscillators are independent. You have to get rid of the coupling term $\lambda x_1 x_2$

Try this variable transformation:

$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$ and $y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$

Then Hamiltonian changes to

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m(\omega^2 + \frac{\lambda}{m})y_1^2 + \frac{1}{2}m(\omega^2 - \frac{\lambda}{m})y_2^2$$

Since $\lambda < m\omega^2$ so we can define two new frequencies: $\omega_1 = \sqrt{\omega^2 + \frac{\lambda}{m}}$ and $\omega_2 = \sqrt{\omega^2 - \frac{\lambda}{m}}$ and the Hamiltonian changes to:

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_1^2 y_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_2^2 y_2^2 = H_1 + H_2$$

Assume $H\psi = E\psi$ where $\psi = \psi_1(y_1)\psi_2(y_2)$ and $E = E_1 + E_2$, you will have

$H_1\psi_1 = E_1\psi_1$ and $H_2\psi_2 = E_2\psi_2$

These becomes two completely independent harmonic oscillators

You will have $E_1 = E_m = \hbar\omega(m + \frac{1}{2})$ and $E_2 = E_n = \hbar\omega(n + \frac{1}{2})$ such that your final energy levels are controlled by two quantum numbers: m and n:

$$E_{m,n} = \hbar\omega(m + n + 1)$$