Homework 6

Problem 2.14

The new allowed energies are $E_n' = (n+\frac{1}{2})\hbar\omega' = 2(n+\frac{1}{2})\hbar\omega = \hbar\omega, 3\hbar\omega, 5\hbar\omega, \ldots$. So the probability of getting $\frac{1}{2}\hbar\omega$ is $\boxed{\text{zero.}}$ The probability of getting $\hbar\omega$ (the new ground state energy) is $P_0 = |c_0|^2$, where $c_0 = \int \Psi(x,0)\psi_0'\,dx$, with

$$\Psi(x,0) = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \quad \psi_0(x)' = \left(\frac{m2\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m2\omega}{2\hbar}x^2}.$$

So

$$c_0 = 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{3m\omega}{2\hbar}x^2}\,dx = 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}}2\sqrt{\pi}\left(\frac{1}{2}\sqrt{\frac{2\hbar}{3m\omega}}\right) = 2^{1/4}\sqrt{\frac{2}{3}}.$$

Therefore

$$P_0 = \boxed{\frac{2}{3}\sqrt{2} = 0.9428.}$$

Problem 2.17

(a)

$$\frac{d}{d\xi}(e^{-\xi^2}) = -2\xi e^{-\xi^2}; \ \left(\frac{d}{d\xi}\right)^2 e^{-\xi^2} = \frac{d}{d\xi}(-2\xi e^{-\xi^2}) = (-2+4\xi^2)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \frac{d}{d\xi}\left[(-2+4\xi^2)e^{-\xi^2}\right] = \left[8\xi + (-2+4\xi^2)(-2\xi)\right]e^{-\xi^2} = (12\xi - 8\xi^3)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \frac{d}{d\xi}\left[(12\xi-8\xi^3)e^{-\xi^2}\right] = \left[12 - 24\xi^2 + (12\xi-8\xi^3)(-2\xi)\right]e^{-\xi^2} = (12 - 48\xi^2 + 16\xi^4)e^{-\xi^2}.$$

$$H_3(\xi) = -e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \boxed{-12\xi + 8\xi^3;}\ H_4(\xi) = e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \boxed{12 - 48\xi^2 + 16\xi^4.}$$

(b)

$$H_5 = 2\xi H_4 - 8H_3 = 2\xi(12 - 48\xi^2 + 16\xi^4) - 8(-12\xi + 8\xi^3) = \boxed{120\xi - 160\xi^3 + 32\xi^5.}$$

$$H_6 = 2\xi H_5 - 10H_4 = 2\xi(120\xi - 160\xi^3 + 32\xi^5) - 10(12 - 48\xi^2 + 16\xi^4) = \boxed{-120 + 720\xi^2 - 480\xi^4 + 64\xi^6.}$$

(c)

$$\frac{dH_5}{d\xi} = 120 - 480\xi^2 + 160\xi^4 = 10(12 - 48\xi^2 + 16\xi^4) = (2)(5)H_4. \checkmark$$

$$\frac{dH_6}{d\xi} = 1440\xi - 1920\xi^3 + 384\xi^5 = 12(120\xi - 160\xi^3 + 32\xi^5) = (2)(6)H_5. \checkmark$$

(d)

$$\frac{d}{dz}(e^{-z^2+2z\xi}) = (-2z+\xi)e^{-z^2+2z\xi}; \text{ setting } z = 0,\ \boxed{H_0(\xi) = 2\xi.}$$

$$\left(\frac{d}{dz}\right)^2 (e^{-z^2+2z\xi}) = \frac{d}{dz}\left[(-2z+2\xi)e^{-z^2+2z\xi}\right]$$
$$= \left[-2+(-2z+2\xi)^2\right]e^{-z^2+2z\xi}; \text{ setting } z=0, \boxed{H_1(\xi) = -2+4\xi^2.}$$

$$\left(\frac{d}{dz}\right)^3 (e^{-z^2+2z\xi}) = \frac{d}{dz}\left\{\left[-2+(-2z+2\xi)^2\right]e^{-z^2+2z\xi}\right\}$$
$$= \left\{2(-2z+2\xi)(-2)+\left[-2+(-2z+2\xi)^2\right](-2z+2\xi)\right\}e^{-z^2+2z\xi};$$

setting $z=0$, $H_2(\xi) = -8\xi + (-2+4\xi^2)(2\xi) = \boxed{-12\xi+8\xi^3.}$

Problem 2.19

Equation 2.94 says $\Psi = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$, so

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\frac{\hbar k^2}{2m}t)}(-ik)e^{-i(kx-\frac{\hbar k^2}{2m}t)} - e^{-i(kx-\frac{\hbar k^2}{2m}t)}(ik)e^{i(kx-\frac{\hbar k^2}{2m}t)}\right]$$
$$= \frac{i\hbar}{2m}|A|^2(-2ik) = \boxed{\frac{\hbar k}{m}|A|^2.}$$

It flows in the positive (x) direction (as you would expect).

Problem 2.21

(a)

$$1 = \int_{-\infty}^{\infty}|\Psi(x,0)|^2dx = 2|A|^2\int_0^{\infty}e^{-2ax}dx = 2|A|^2\left.\frac{e^{-2ax}}{-2a}\right|_0^{\infty} = \frac{|A|^2}{a} \Rightarrow A = \boxed{\sqrt{a}.}$$

(b)

$$\phi(k) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}e^{-ikx}\,dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}(\cos kx - i\sin kx)dx.$$

The cosine integrand is even, and the sine is odd, so the latter vanishes and

$$\phi(k) = 2\frac{A}{\sqrt{2\pi}}\int_0^{\infty}e^{-ax}\cos kx\,dx = \frac{A}{\sqrt{2\pi}}\int_0^{\infty}e^{-ax}\left(e^{ikx}+e^{-ikx}\right)\,dx$$
$$= \frac{A}{\sqrt{2\pi}}\int_0^{\infty}\left(e^{(ik-a)x}+e^{-(ik+a)x}\right)dx = \frac{A}{\sqrt{2\pi}}\left.\left[\frac{e^{(ik-a)x}}{ik-a}+\frac{e^{-(ik+a)x}}{-(ik+a)}\right]\right|_0^{\infty}$$
$$= \frac{A}{\sqrt{2\pi}}\left(\frac{-1}{ik-a}+\frac{1}{ik+a}\right) = \frac{A}{\sqrt{2\pi}}\frac{-ik-a+ik-a}{-k^2-a^2} = \boxed{\sqrt{\frac{a}{2\pi}}\frac{2a}{k^2+a^2}.}$$

(c)

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} 2\sqrt{\frac{a^3}{2\pi}} \int_{-\infty}^{\infty} \frac{1}{k^2+a^2} e^{i(kx - \frac{\hbar k^2}{2m}t)} dk = \boxed{\frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+a^2} e^{i(kx - \frac{\hbar k^2}{2m}t)} dk.}$$

(d) For *large* a, $\Psi(x,0)$ is a sharp narrow spike whereas $\phi(k) \cong \sqrt{2/\pi a}$ is broad and flat; position is well-defined but momentum is ill-defined. For *small* a, $\Psi(x,0)$ is a broad and flat whereas $\phi(k) \cong (\sqrt{2a^3/\pi})/k^2$ is a sharp narrow spike; position is ill-defined but momentum is well-defined.