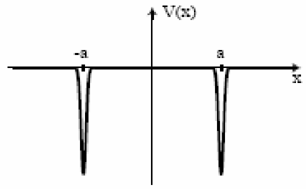
Homework 7

Problem 2.27

(a)



(b) From Problem 2.1(c) the solutions are even or odd. Look first for *even solutions*:

$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} + e^{-\kappa x}) & (-a < x < a), \\ Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} + e^{-\kappa a})$, or $A = B(e^{2\kappa a} + 1)$.

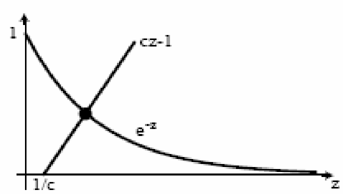
Discontinuous derivative at a, $\Delta \frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2}\psi(a)$:

$$-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} - \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2} A e^{-\kappa a} \Rightarrow A + B(e^{2\kappa a} - 1) = \frac{2m\alpha}{\hbar^2 \kappa} A; \text{ or}$$

$$B(e^{2\kappa a} - 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = B(e^{2\kappa a} + 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) \Rightarrow e^{2\kappa a} - 1 = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) + \frac{2m\alpha}{\hbar^2\kappa} - 1.$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 + \frac{2m\alpha}{\hbar^2\kappa} e^{-2\kappa a}; \quad \frac{\hbar^2\kappa}{m\alpha} = 1 + e^{-2\kappa a}, \text{ or } \boxed{e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha} - 1.}$$

This is a transcendental equation for κ (and hence for E). I'll solve it graphically: Let $z \equiv 2\kappa a$, $c \equiv \frac{\hbar^2}{2am\alpha}$, so $e^{-z} = cz - 1$. Plot both sides and look for intersections:



From the graph, noting that c and z are both positive, we see that there is one (and only one) solution (for even ψ). If $\alpha = \frac{\hbar^2}{2ma}$, so $c = 1$, the calculator gives $z = 1.278$, so $\kappa^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \Rightarrow E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right) = -0.204\left(\frac{\hbar^2}{ma^2}\right)$.

Now look for *odd solutions:*

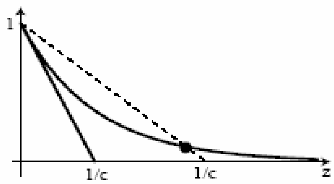
$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} - e^{-\kappa x}) & (-a < x < a), \\ -Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} - e^{-\kappa a})$, or $A = B(e^{2\kappa a} - 1)$.

Discontinuity in ψ': $-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} + \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2} A e^{-\kappa a} \Rightarrow B(e^{2\kappa a} + 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right)$,

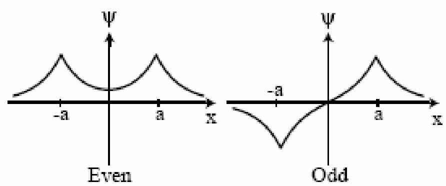
$$e^{2\kappa a} + 1 = (e^{2\kappa a} - 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) - \frac{2m\alpha}{\hbar^2\kappa} + 1,$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 - \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a};\ \frac{\hbar^2\kappa}{m\alpha} = 1 - e^{-2\kappa a},\ \boxed{e^{-2\kappa a} = 1 - \frac{\hbar^2\kappa}{m\alpha},}\ \text{or } e^{-z} = 1 - cz.$$



This time there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (solid line), whereas if c is smaller (dashed line) there will be. (Note that $z = 0 \Rightarrow \kappa = 0$ is *not* a solution, since ψ is then non-normalizable.) The slope of e^{-z} (at $z = 0$) is -1; the slope of $(1 - cz)$ is $-c$. So there is an *odd* solution $\Leftrightarrow c < 1$, or $\alpha > \hbar^2/2ma$.

Conclusion: $\boxed{\textit{One}\text{ bound state if } \alpha \le \hbar^2/2ma;\ \textit{two}\text{ if } \alpha > \hbar^2/2ma.}$



$$\alpha = \frac{\hbar^2}{ma} \Rightarrow c = \frac{1}{2}.\ \begin{cases} \textit{Even: } e^{-z} = \frac{1}{2}z - 1 \Rightarrow z = 2.21772, \\ \textit{Odd: } \ e^{-z} = 1 - \frac{1}{2}z \Rightarrow z = 1.59362. \end{cases}$$

$$\boxed{E = -0.615(\hbar^2/ma^2);\ E = -0.317(\hbar^2/ma^2).}$$

$$\alpha = \frac{\hbar^2}{4ma} \Rightarrow c = 2.\ \textit{Only even: } e^{-z} = 2z - 1 \Rightarrow z = 0.738835;\ \boxed{E = -0.0682(\hbar^2/ma^2).}$$

Problem 2.29

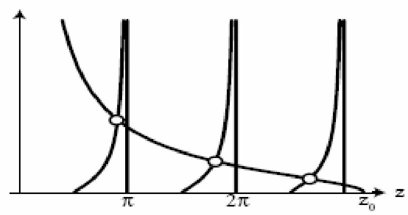
In place of Eq. 2.151, we have: $\psi(x) = \left\{ \begin{array}{ll} Fe^{-\kappa x} & (x > a) \\ D\sin(lx) & (0 < x < a) \\ -\psi(-x) & (x < 0) \end{array} \right\}.$

Continuity of ψ : $Fe^{-\kappa a} = D\sin(la)$; continuity of ψ' : $-F\kappa e^{-\kappa a} = Dl\cos(la)$.

Divide: $-\kappa = l\cot(la)$, or $-\kappa a = la\cot(la) \Rightarrow \sqrt{z_0^2 - z^2} = -z\cot z$, or $\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}.}$

Wide, deep well: Intersections are at $\pi, 2\pi, 3\pi$, etc. Same as Eq. 2.157, but now for n *even*. This fills in the rest of the states for the infinite square well.

Shallow, narrow well: If $z_0 < \pi/2$, there is *no* odd bound state. The corresponding condition on V_0 is $\boxed{V_0 < \dfrac{\pi^2\hbar^2}{8ma^2} \Rightarrow \textit{no} \text{ odd bound state.}}$



Problem 2.34

(a)

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

(b)

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{array} \right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ : $A + B = F$.

(2) Continuity of ψ' : $ik(A - B) = ilF$.

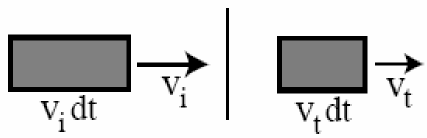
$$\Rightarrow A + B = \frac{k}{l}(A - B); \; A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1 - k/l)^2}{(1 + k/l)^2} = \frac{(k-l)^2}{(k+l)^2} = \frac{(k-l)^4}{(k^2 - l^2)^2}.$$

Now $k^2 - l^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \left(\frac{2m}{\hbar^2}\right)V_0$; $k - l = \frac{\sqrt{2m}}{\hbar}[\sqrt{E} - \sqrt{E - V_0}]$, so

$$\boxed{R = \frac{(\sqrt{E} - \sqrt{E - V_0})^4}{V_0^2}.}$$

(c)



From the diagram, $T = P_t/P_i = |F|^2 v_t/|A|^2 v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\frac{v_t}{v_i} = \frac{\sqrt{E - V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T = \sqrt{\frac{E - V_0}{E}}\left|\frac{F}{A}\right|^2$. Alternatively, from Problem 2.19:

$$J_i = \frac{\hbar k}{m}|A|^2; \quad J_t = \frac{\hbar l}{m}|F|^2; \quad T = \frac{J_t}{J_i} = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left|\frac{F}{A}\right|^2 \sqrt{\frac{E - V_0}{E}}.$$

For $E < V_0$, of course, $\boxed{T = 0.}$

(d)

For $E > V_0$, $F = A + B = A + A\frac{(\frac{k}{l} - 1)}{(\frac{k}{l} + 1)} = A\frac{2k/l}{(\frac{k}{l} + 1)} = \frac{2k}{k + l}A.$

$$T = \left|\frac{F}{A}\right|^2 \frac{l}{k} = \left(\frac{2k}{k+l}\right)^2 \frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2 - l^2)^2} = \boxed{\frac{4\sqrt{E}\sqrt{E - V_0}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}.}$$

$$T + R = \frac{4kl}{(k+l)^2} + \frac{(k-l)^2}{(k+l)^2} = \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} = \frac{k^2 + 2kl + l^2}{(k+l)^2} = \frac{(k+l)^2}{(k+l)^2} = 1. \checkmark$$

Problem 2.40

(a) Let $V_0 \equiv 32\hbar^2/ma^2$. This is just like the *odd* bound states for the finite square well, since they are the ones that go to zero at the origin. Referring to the solution to Problem 2.29, the wave function is

$$\psi(x) = \begin{cases} D \sin lx, & l \equiv \sqrt{2m(E+V_0)}/\hbar \quad (0 < x < a), \\ Fe^{-\kappa x}, & \kappa \equiv \sqrt{-2mE}/\hbar \quad (x > a), \end{cases}$$

and the boundary conditions at $x = a$ yield

$$-\cot z = \sqrt{(z_0/z)^2 - 1}$$

with

$$z_0 = \frac{\sqrt{2mV_0}}{\hbar}a = \frac{\sqrt{2m(32\hbar^2/ma^2)}}{\hbar}a = 8.$$

Referring to the figure (Problem 2.29), and noting that $(5/2)\pi = 7.85 < z_0 < 3\pi = 9.42$, we see that there are $\boxed{\text{three bound states.}}$

(b) Let

$$I_1 \equiv \int_0^a |\psi|^2 dx = |D|^2 \int_0^a \sin^2 lx \, dx = |D|^2 \left[\frac{x}{2} - \frac{1}{2l}\sin lx \cos lx\right]\Bigg|_0^a = |D|^2 \left[\frac{a}{2} - \frac{1}{2l}\sin lz \cos la\right];$$

$$I_2 \equiv \int_a^\infty |\psi|^2 dx = |F|^2 \int_a^\infty e^{-2\kappa x} \, dx = |F|^2 \left[-\frac{e^{-2\kappa x}}{2\kappa}\right]\Bigg|_a^\infty = |F|^2 \frac{e^{-2\kappa a}}{2\kappa}.$$

But continuity at $x = a \Rightarrow Fe^{-\kappa a} = D \sin la$, so $I_2 = |D|^2 \frac{\sin^2 la}{2\kappa}$.

Normalizing:

$$1 = I_1 + I_2 = |D|^2 \left[\frac{a}{2} - \frac{1}{2l}\sin la \cos la + \frac{\sin^2 la}{2\kappa}\right] = \frac{1}{2\kappa}|D|^2 \left[\kappa a - \frac{\kappa}{l}\sin la \cos la + \sin^2 la\right]$$

But (referring again to Problem 2.29) $\kappa/l = -\cot la$, so

$$= \frac{1}{2\kappa}|D|^2 \left[\kappa a + \cot la \sin la \cos la + \sin^2 la\right] = |D|^2 \frac{(1+\kappa a)}{2\kappa}.$$

So $|D|^2 = 2\kappa/(1+\kappa a)$, and the probability of finding the particle outside the well is

$$P = I_2 = \frac{2\kappa}{1+\kappa a}\frac{\sin^2 la}{2\kappa} = \frac{\sin^2 la}{1+\kappa a}.$$

We can express this interms of $z \equiv la$ and z_0: $\kappa a = \sqrt{z_0^2 - z^2}$ (page 80),

$$\sin^2 la = \sin^2 z = \frac{1}{1+\cot^2 z} = \frac{1}{1+(z_0/z)^2 - 1} = \left(\frac{z}{z_0}\right)^2 \Rightarrow P = \frac{z^2}{z_0^2(1+\sqrt{z_0^2 - z^2})}.$$

So far, this is correct for *any* bound state. In the present case $z_0 = 8$ and z is the third solution to $-\cot z = \sqrt{(8/z)^2 - 1}$, which occurs somewhere in the interval $7.85 < z < 8$. Mathematica gives $z = 7.9573$ and $\boxed{P = 0.54204.}$

```
FindRoot[Cot[z] == -√((8/z)^2 - 1), {z, 7.9}]

{z → 7.95732}

z^2/(64 (1 + √(64 - z^2)))

       z²
64 (1 + √(64 - z²))

% /. z -> 7.957321523328964`

0.542041
```