Homework 8

## Problem 2.38

(a) New allowed energies: $E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$; $\Psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)$, $\psi_n(x) = \sqrt{\frac{2}{2a}}\sin\left(\frac{n\pi}{2a}x\right)$.

$$c_n = \frac{\sqrt{2}}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{n\pi}{2a}x\right)dx = \frac{\sqrt{2}}{2a}\int_0^a \left\{\cos\left[\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right] - \cos\left[\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right]\right\}dx.$$

$$= \frac{1}{\sqrt{2}a}\left\{\frac{\sin\left[\left(\frac{n}{2}-1\right)\frac{\pi x}{a}\right]}{\left(\frac{n}{2}-1\right)\frac{\pi}{a}} - \frac{\sin\left[\left(\frac{n}{2}+1\right)\frac{\pi x}{a}\right]}{\left(\frac{n}{2}+1\right)\frac{\pi}{a}}\right\}\Bigg|_0^a \quad (\text{for } n \neq 2)$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{\sin\left[\left(\frac{n}{2}-1\right)\pi\right]}{\left(\frac{n}{2}-1\right)} - \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{\left(\frac{n}{2}+1\right)}\right\} = \frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{\sqrt{2}\pi}\left[\frac{1}{\left(\frac{n}{2}-1\right)} - \frac{1}{\left(\frac{n}{2}+1\right)}\right]$$

$$= \frac{4\sqrt{2}}{\pi}\frac{\sin\left[\left(\frac{n}{2}+1\right)\pi\right]}{(n^2-4)} = \left\{\begin{array}{ll} 0, & \text{if } n \text{ is even} \\ \pm\frac{4\sqrt{2}}{\pi(n^2-4)}, & \text{if } n \text{ is odd}\end{array}\right\}.$$

$$c_2 = \frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{\sqrt{2}}{a}\int_0^a \frac{1}{2}dx = \frac{1}{\sqrt{2}}.$$ So the probability of getting $E_n$ is

$$P_n = |c_n|^2 = \left\{\begin{array}{ll} \frac{1}{2}, & \text{if } n = 2 \\ \frac{32}{\pi^2(n^2-4)^2}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise}\end{array}\right\}.$$

Most probable: $E_2 = \boxed{\frac{\pi^2\hbar^2}{2ma^2}}$ (same as before). Probability: $P_2 = \boxed{1/2.}$

(b) Next most probable: $E_1 = \boxed{\frac{\pi^2\hbar^2}{8ma^2}}$, with probability $P_1 = \boxed{\frac{32}{9\pi^2} = 0.36025.}$

(c) $\langle H\rangle = \int \Psi^* H \Psi\, dx = \frac{2}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\sin\left(\frac{\pi}{a}x\right)dx$, but this is exactly the same as before the wall moved – for which we know the answer: $\boxed{\frac{\pi^2\hbar^2}{2ma^2}}$.

## Problem 2.46

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$ (where $x$ is measured around the circumference), or $\frac{d^2\psi}{dx^2} = -k^2\psi$, with $k \equiv \frac{\sqrt{2mE}}{\hbar}$, so

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

But $\psi(x+L) = \psi(x)$, since $x+L$ is the same point as $x$, so

$$Ae^{ikx}e^{ikL} + Be^{-ikx}e^{-ikL} = Ae^{ikx} + Be^{-ikx},$$

and this is true for *all* $x$. In particular, for $x = 0$ :

(1) $Ae^{ikL} + Be^{-ikL} = A + B$. And for $x = \frac{\pi}{2k}$ :

$Ae^{i\pi/2}e^{ikL} + Be^{-i\pi/2}e^{-ikL} = Ae^{i\pi/2} + Be^{-i\pi/2}$, or $iAe^{ikL} - iBe^{-ikL} = iA - iB$, so

(2) $Ae^{ikL} - Be^{-ikL} = A - B$. Add (1) and (2): $2Ae^{ikL} = 2A$.

Either $A = 0$, or else $e^{ikL} = 1$, in which case $kL = 2n\pi$ ($n = 0, \pm1, \pm2, \ldots$). But if $A = 0$, then $Be^{-ikL} = B$, leading to the same conclusion. So for every positive $n$ there are *two* solutions: $\psi_n^+(x) = Ae^{i(2n\pi x/L)}$ and $\psi_n^-(x) = Be^{-i(2n\pi x/L)}$ ($n = 0$ is ok too, but in that case there is just *one* solution). Normalizing: $\int_0^L |\psi_\pm|^2 dx = 1 \Rightarrow A = B = 1/\sqrt{L}$. Any *other* solution (with the same energy) is a linear combination of these.

$$\boxed{\psi_n^\pm(x) = \frac{1}{\sqrt{L}}e^{\pm i(2n\pi x/L)};\ E_n = \frac{2n^2\pi^2\hbar^2}{mL^2}\ (n = 0, 1, 2, 3, \ldots).}$$

The theorem fails because here $\psi$ does *not* go to zero at $\infty$; $x$ is restricted to a finite range, and we are unable to determine the constant $K$ (in Problem 2.45).

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## Problem 2.49

(a)

$$\frac{\partial\Psi}{\partial t} = \left(-\frac{m\omega}{2\hbar}\right)\left[\frac{a^2}{2}\left(-2i\omega e^{-2i\omega t}\right) + \frac{i\hbar}{m} - 2ax(-i\omega)e^{-i\omega t}\right]\Psi, \text{ so}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{1}{2}ma^2\omega^2 e^{-2i\omega t} + \frac{1}{2}\hbar\omega + max\omega^2 e^{-i\omega t}\right]\Psi.$$

$$\frac{\partial\Psi}{\partial x} = \left[\left(-\frac{m\omega}{2\hbar}\right)\left(2x - 2ae^{-i\omega t}\right)\right]\Psi = -\frac{m\omega}{\hbar}\left(x - ae^{-i\omega t}\right)\Psi;$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{m\omega}{\hbar}\Psi - \frac{m\omega}{\hbar}\left(x - ae^{-i\omega t}\right)\frac{\partial\Psi}{\partial x} = \left[-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2\left(x - ae^{-i\omega t}\right)^2\right]\Psi.$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\Psi = -\frac{\hbar^2}{2m}\left[-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2\left(x-ae^{-i\omega t}\right)^2\right]\Psi+\frac{1}{2}m\omega^2x^2\Psi$$
$$= \left[\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2\left(x^2-2axe^{-i\omega t}+a^2e^{-2i\omega t}\right)+\frac{1}{2}m\omega^2x^2\right]\Psi$$
$$= \left[\frac{1}{2}\hbar\omega+max\omega^2e^{-i\omega t}-\frac{1}{2}m\omega^2a^2e^{-2i\omega t}\right]\Psi$$
$$= i\hbar\frac{\partial\Psi}{\partial t}\text{ (comparing second line above).} \quad ✓$$

(b)

$$|\Psi|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{2\hbar}\left[\left(x^2+\frac{a^2}{2}(1+e^{2i\omega t})-\frac{i\hbar t}{m}-2axe^{i\omega t}\right)+\left(x^2+\frac{a^2}{2}(1+e^{-2i\omega t})+\frac{i\hbar t}{m}-2axe^{-i\omega t}\right)\right]}$$
$$= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{2\hbar}\left[2x^2+a^2+a^2\cos(2\omega t)-4ax\cos(\omega t)\right]}.\quad \text{But } a^2[1+\cos(2\omega t)] = 2a^2\cos^2\omega t, \text{ so}$$
$$= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}\left[x^2-2ax\cos(\omega t)+a^2\cos^2(\omega t)\right]} = \boxed{\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}(x-a\cos\omega t)^2}.}$$

The wave packet is a *Gaussian* of fixed shape, whose *center* oscillates back and forth sinusoidally, with amplitude $a$ and angular frequency $\omega$.

(c) Note that this wave function *is* correctly normalized (compare Eq. 2.59). Let $y \equiv x - a\cos\omega t$:

$$\langle x\rangle = \int x|\Psi|^2dx = \int(y+a\cos\omega t)|\Psi|^2dy = 0 + a\cos\omega t\int|\Psi|^2dy = \boxed{a\cos\omega t.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{-ma\omega\sin\omega t.}\quad \frac{d\langle p\rangle}{dt} = -ma\omega^2\cos\omega t.\quad V = \frac{1}{2}m\omega^2x^2 \Longrightarrow \frac{dV}{dx} = m\omega^2x.$$

$$\langle -\frac{dV}{dx}\rangle = -m\omega^2\langle x\rangle = -m\omega^2a\cos\omega t = \frac{d\langle p\rangle}{dt}, \text{ so Ehrenfest's theorem } \textit{is} \text{ satisfied.}$$

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ADDITIONAL PROBLEM SOLUTION:

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\delta(x-a)\psi = E\psi$$

For a bound state, E < 0, so that we assume that

$$\beta = \sqrt{-2mE}/\hbar$$

and the Schrödinger equation becomes

$$\psi'' - \beta^2\psi + \frac{2mV_0}{\hbar^2}\delta(x-a) = 0 \quad \text{--------------(1)}$$

The "jump" condition at x = a is given by

$$\psi'(a^+) - \psi'(a^-) = -\frac{2mV_0}{\hbar^2}\psi(a) \quad \text{--------------(2)}$$

For x ≠ a the equation (1) is simplified as

$\psi'' - \beta^2\psi = 0$ -------------- (3)

And the boundary conditions are $\psi(0) = 0$ and $\psi(\infty) = 0$ such that

$\psi(x) = sh(\beta x)$ for $0 \le x < a$

$\psi(x) = Ae^{-\beta x}$ for $x > a$

The continuity at x = a and the "jump" condition (eq (2)) give

$$sh(\beta a) = Ae^{-\beta a} = \psi(a)$$

$$-\beta Ae^{-\beta a} - \beta ch(\beta a) = -\frac{2mV_0}{\hbar^2}\psi(a)$$

Or $\beta a[1 + \coth(\beta a)] = \frac{2mV_0}{\hbar^2}$ --------- (4)

Eq (4) determines the energy levels.

For the first energy level, $E \approx 0$ or $\beta a \approx 0^+$. Using

$$\lim_{\beta a \to 0} \beta a \coth(\beta a) = \lim_{\beta a \to 0} \frac{\beta a}{\tanh(\beta a)} = 1$$

Eq. (4) becomes

$$\frac{2mV_0 a}{\hbar^2} = \beta a[1 + \coth(\beta a)] = 1 + 0^+$$

So that the condition for having at least one bound state is

$$\frac{2mV_0 a}{\hbar^2} \ge 0^+$$

This is unlike the case of a single δ-potential well in which at least a bound state exists, the existence of the infinite potential wall at x = 0 give rise to a "repulsive" effect on the particle such that "at least one bound state" is not guarantied any more.