Homework 9: **Chapter 3**

Problem 3.5

(a) $\langle f|xg\rangle = \int f^*(xg)\,dx = \int (xf)^* g\,dx = \langle xf|g\rangle$, so $\boxed{x^\dagger = x.}$

$\langle f|ig\rangle = \int f^*(ig)\,dx = \int (-if)^* g\,dx = \langle -if|g\rangle$, so $\boxed{i^\dagger = -i.}$

$$\langle f|\frac{dg}{dx}\rangle = \int_{-\infty}^{\infty} f^* \frac{dg}{dx}\,dx = f^* g\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{df}{dx}\right)^* g\,dx = -\langle xf|g\rangle, \text{ so } \boxed{\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}.}$$

(b) $a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$. But p and x are hermitian, and $i^\dagger = -i$, so $(a_+)^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x)$, or $\boxed{(a_+)^\dagger = (a_-).}$

(c) $\langle f|(\hat{Q}\hat{R})g\rangle = \langle \hat{Q}^\dagger f|\hat{R}g\rangle = \langle \hat{R}^\dagger\hat{Q}^\dagger f|g\rangle = \langle(\hat{Q}\hat{R})^\dagger f|g\rangle$, so $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$. ✓

Problem 3.13

(a) $[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B$. ✓

(b) Introducing a test function $g(x)$, as in Eq. 2.50:

$$[x^n, p]g = x^n \frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(x^n g) = x^n \frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(nx^{n-1}g + x^n\frac{dg}{dx}\right) = i\hbar n x^{n-1} g.$$

So, dropping the test function, $[x^n, p] = i\hbar n x^{n-1}$. ✓

(c) $[f,p]g = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(fg) = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(\frac{df}{dx}g + f\frac{dg}{dx}\right) = i\hbar\frac{df}{dx}g \Rightarrow [f,p] = i\hbar\frac{df}{dx}$. ✓

Problem 3.26

(a) $\langle Q\rangle = \langle\psi|\hat{Q}\psi\rangle = \langle\hat{Q}^\dagger\psi|\psi\rangle = -\langle\hat{Q}\psi|\psi\rangle = -(\langle\psi|\hat{Q}\psi\rangle)^* = -\langle Q\rangle^*$, so $\langle Q\rangle$ is imaginary. ✓

(b) From Problem 3.5(c) we know that $(\hat{P}\hat{Q})^\dagger = \hat{Q}^\dagger\hat{P}^\dagger$, so if $\hat{P} = \hat{P}^\dagger$ and $\hat{Q} = \hat{Q}^\dagger$ then

$$[\hat{P},\hat{Q}]^\dagger = (\hat{P}\hat{Q} - \hat{Q}\hat{P})^\dagger = \hat{Q}^\dagger\hat{P}^\dagger - \hat{P}^\dagger\hat{Q}^\dagger = \hat{Q}\hat{P} - \hat{P}\hat{Q} = -[\hat{P},\hat{Q}]. \quad ✓$$

If $\hat{P} = -\hat{P}^\dagger$ and $\hat{Q} = -\hat{Q}^\dagger$, then $[\hat{P},\hat{Q}]^\dagger = \hat{Q}^\dagger\hat{P}^\dagger - \hat{P}^\dagger\hat{Q}^\dagger = (-\hat{Q})(-\hat{P}) - (-\hat{P})(-\hat{Q}) = -[\hat{P},\hat{Q}]$.

So in either case the commutator is antihermitian.

Problem 3.31

Equation 3.71 $\Rightarrow \frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle [H, xp]\rangle$; Eq. 3.64 $\Rightarrow [H, xp] = [H, x]p + x[H, p]$; Problem 3.14 $\Rightarrow [H, x] = -\frac{i\hbar p}{m}$; Problem 3.17(d) $\Rightarrow [H, p] = i\hbar\frac{dV}{dx}$. So

$$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\frac{dV}{dx}\rangle\right] = 2\langle\frac{p^2}{2m}\rangle - \langle x\frac{dV}{dx}\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle. \quad \text{QED}$$

In a stationary state all expectation values (at least, for operators that do not depend explicitly on t) are time-independent (see item 1 on p. 26), so $d\langle xp\rangle/dt = 0$, and we are left with Eq. 3.97.

For the harmonic oscillator:

$$V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{dV}{dx} = m\omega^2 x \Rightarrow x\frac{dV}{dx} = m\omega^2x^2 = 2V \Rightarrow 2\langle T\rangle = 2\langle V\rangle \Rightarrow \langle T\rangle = \langle V\rangle. \quad \text{QED}$$

In Problem 2.11(c) we found that $\langle T\rangle = \langle V\rangle = \frac{1}{4}\hbar\omega$ (for $n=0$); $\langle T\rangle = \langle V\rangle = \frac{3}{4}\hbar\omega$ (for $n=1$). ✓
In Problem 2.12 we found that $\langle T\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$, while $\langle x^2\rangle = (n+\frac{1}{2})\hbar/m\omega$, so $\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$, and hence $\langle T\rangle = \langle V\rangle$ for *all* stationary states. ✓

Problem 3.36

(a) Equation 3.60 becomes $|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = \left[\frac{1}{2}(z+z^*)\right]^2 + \left[\frac{1}{2i}(z-z^*)\right]^2$; Eq. 3.61 generalizes to

$$\sigma_A^2\sigma_B^2 \geq \left[\frac{1}{2}\left(\langle f|g\rangle + \langle g|f\rangle\right)\right]^2 + \left[\frac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)\right]^2.$$

But $\langle f|g\rangle - \langle g|f\rangle = \langle[\hat{A},\hat{B}]\rangle$ (p. 111), and, by the same argument,

$$\langle f|g\rangle + \langle g|f\rangle = \langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle + \langle\hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle = \langle\hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle A\rangle\langle B\rangle\rangle = \langle D\rangle.$$

So $\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\left(\langle D\rangle^2 + \langle C\rangle^2\right)$. ✓

(b) If $\hat{B} = \hat{A}$, then $\hat{C} = 0$, $\hat{D} = 2\left(\hat{A}^2 - \langle A\rangle^2\right)$; $\langle D\rangle = 2\left(\langle\hat{A}^2\rangle - \langle A\rangle^2\right) = 2\sigma_A^2$. So Eq. 3.99 says $\sigma_A^2\sigma_A^2 \geq (1/4)4\sigma_A^4 = \sigma_A^4$, which is *true*, but not very informative.

Problem 3.39

(a)

Expanding in a Taylor series: $f(x+x_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\,x_0^n\left(\frac{d}{dx}\right)^n f(x).$

But $p=\frac{\hbar}{i}\frac{d}{dx}$, so $\frac{d}{dx}=\frac{ip}{\hbar}$. Therefore $f(x+x_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\,x_0^n\left(\frac{ip}{\hbar}\right)^n f(x)=e^{ipx_0/\hbar}f(x).$

(b)

$$\Psi(x,t+t_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\,t_0^n\left(\frac{\partial}{\partial t}\right)^n\Psi(x,t);\quad i\hbar\frac{\partial\Psi}{\partial t}=H\Psi.$$

[*Note:* It is emphatically *not* the case that $i\hbar\frac{\partial}{\partial t}=H$. These two operators have the same effect *only* when (as here) they are acting on solutions to the (time-dependent) Schrödinger equation.] Also,

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2\Psi=i\hbar\frac{\partial}{\partial t}(H\Psi)=H\left(i\hbar\frac{\partial\Psi}{\partial t}\right)=H^2\Psi,$$

provided H is not explicitly dependent on t. And so on. So

$$\Psi(x,t+t_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\,t_0^n\left(-\frac{i}{\hbar}H\right)^n\Psi=e^{-iHt_0/\hbar}\Psi(x,t).$$

(c)

$$\langle Q\rangle_{t+t_0}=\langle\Psi(x,t+t_0)|Q(x,p,t+t_0)|\Psi(x,t+t_0)\rangle.$$

But $\Psi(x,t+t_0)=e^{-iHt_0/\hbar}\Psi(x,t)$, so, using the hermiticity of H to write $\left(e^{-iHt_0/\hbar}\right)^\dagger=e^{iHt_0/\hbar}$:

$$\langle Q\rangle_{t+t_0}=\langle\Psi(x,t)|e^{iHt_0/\hbar}Q(x,p,t+t_0)e^{-iHt_0/\hbar}|\Psi(x,t)\rangle.$$

If $t_0=dt$ is very small, expanding to first order, we have:

$$\langle Q\rangle_t+\frac{d\langle Q\rangle}{dt}dt=\langle\Psi(x,t)|\underbrace{\left(1+\frac{iH}{\hbar}dt\right)\left[Q(x,p,t)+\frac{\partial Q}{\partial t}dt\right]\left(1-\frac{iH}{\hbar}dt\right)}_{\bigstar}|\Psi(x,t)\rangle$$

$$\left[\bigstar=Q(x,p,t)+\frac{iH}{\hbar}dt\,Q-Q\left(\frac{iH}{\hbar}dt\right)+\frac{\partial Q}{\partial t}dt=Q+\frac{i}{\hbar}[H,Q]dt+\frac{\partial Q}{\partial t}dt\right]$$

$$=\langle Q\rangle_t+\frac{i}{\hbar}\langle[H,Q]\rangle dt+\langle\frac{\partial Q}{\partial t}\rangle dt.$$

$$\therefore\ \frac{d\langle Q\rangle}{dt}=\frac{i}{\hbar}\langle[H,Q]\rangle+\langle\frac{\partial Q}{\partial t}\rangle.\quad\text{QED}$$