1. To understand atomic energy levels, de Broglie hypothesized that all allowed orbits for an electron in a hydrogen atom are given by the condition that an integral number \( n \) de Broglie wavelength fit around the circumference of the orbit. Show that this condition leads to the Bohr quantization condition on angular momentum \( L = n\hbar \).

Solution:

Let the electron have speed \( v \) in an orbit of radius \( r \). The circumference of the orbit is the \( 2\pi r \). de Broglie’s hypothesis is then that an integral number of wave lengths fit into this circumference:

\[
\frac{2\pi r}{\lambda} = n .
\]

Setting \( \lambda = \frac{\hbar}{mv} \) gives

\[
\frac{2\pi m_v v r}{\hbar} = n .
\]

However, the angular momentum of the electron is given by \( L = m_v v r \) such that

\[
L = \left( \frac{\hbar}{2\pi} \right) n = \hbar n ,
\]

precisely the Bohr quantization condition of angular momentum.

2. If \( \Delta x \Delta p = \hbar/2 \) where \( (\Delta x)^2 = <x^2> - <x>^2 \) and \( (\Delta p)^2 = <p^2> - <p>^2 \) but \( <x> = <p> = 0 \), show that the minimum energy of a quantum harmonic oscillator is \( \hbar \omega/2 \), if the Hamiltonian is

\[
H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2
\]

Solution:

Since \( <x> = <p> = 0 \), \( <\Delta x^2> = <x^2> \) and \( <\Delta p^2> = <p^2> \)

\[
< H > = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} x^2 = \frac{1}{2m} [ <\Delta p^2> + m^2 \omega^2 <\Delta x^2> ]
\]

For any real value of \( a \) and \( b \), \( a + b \geq 2\sqrt{ab} \). If \( a = <\Delta p^2> \) and \( b = m^2 \omega^2 <\Delta x^2> \), then

\[
<\Delta p^2> + m^2 \omega^2 <\Delta x^2> \geq 2\sqrt{<\Delta p^2> \cdot m^2 \omega^2 <\Delta x^2>^2} = 2m\omega <\Delta x> <\Delta p> \geq 2m\omega \frac{\hbar}{2}
\]

so that

\[
< H >_{\text{min}} = \frac{1}{2} \hbar \omega
\]
3. Presume that the wave function \( \psi(x) = Ae^{-kx} \) applies over the domain \( 0 \leq x \leq \infty \) (assume that \( \psi = 0 \) outside this domain). What must be the value of A in terms of \( k \) if this function is to be properly normalized? What is the probability of finding the particle between \( x = 0 \) and \( x = 1/k \)?

**Solution:**

We must have \( \int |\psi|^2 \, dx = 1 \). In the present case this demands

\[
A^2 \int_0^\infty e^{-2kx} \, dx = 1
\]

such that \( A = \sqrt{2k} \)

The probability of finding the particle between \( x = 0 \) and \( x = 1/k \) is given by the integral of \( \psi^2 \) over these limits

\[
P(0,1/k) = \int_0^{1/k} |\psi|^2 \, dx = 2k \left[ -\frac{1}{2k} e^{-2kx} \right]_0^{1/k} = 1 - e^{-2} \sim 0.865
\]

4. A particle with mass of \( m \) is confined in a one dimensional infinitive well \( (0 \leq x \leq a) \)

i.e. \( V(x) = \begin{cases} 0 & \text{when } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases} \). And the energy eigenvalues and eigenfunctions are \( E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \) and \( \psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \) with \( n = 1, 2, 3, \ldots \), respectively. At \( t = 0 \), the initial normalized wave function is \( \psi(x,0) = \sqrt{\frac{8}{5a}}(1 + \cos \frac{\pi x}{a}) \sin \frac{\pi x}{a} \). a) Find the wave function at any moment \( t \); b) What is the expectation value of energy; c) Find the probability in the left half of the well \( (0 \leq x \leq a/2) \) at any moment \( t \).

**Solution:**

(a) The initial state can be decomposed as the combination of first two eigenstates:

\[\psi(x,0) = \sqrt{\frac{8}{5a}}(1 + \cos \frac{\pi x}{a}) \sin \frac{\pi x}{a} = \frac{4}{5} \psi_1(x) + \frac{1}{5} \psi_2(x)\]

with

\[\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \quad (n = 1) \text{ and } \psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) \quad (n = 2)\]

According to the superposition principle: \( \psi(x,t) = \sum_n C_n \psi_n(x) e^{-\frac{iE_n t}{\hbar}} \), we have

\[
\psi(x,t) = \frac{4}{5} \psi_1(x) \exp\left(-\frac{ih\pi^2 t}{2ma^2}\right) + \frac{1}{5} \psi_2(x) \exp\left(-\frac{2ih\pi^2 t}{ma^2}\right)
\]

\[
= \sqrt{\frac{8}{5a}} \sin \frac{\pi x}{a} \left[ \exp\left(-\frac{ih\pi^2 t}{2ma^2}\right) + 2 \cos \frac{\pi x}{a} \exp\left(-\frac{2ih\pi^2 t}{ma^2}\right) \right]
\]
(b) Since the energy does not change with time (stationary states), the energy at any moment is given by the initial energy as

$$< H > = E_0 = \frac{4}{5} E_1 + \frac{1}{5} E_2 = \frac{4\hbar^2 \pi^2}{5ma^2}$$

(c) The probability at any moment $t$ is given by

$$P(t)(0 \leq x \leq \frac{a}{2}) = \int_0^{a/2} dx |\psi(x,t)|^2$$

$$= \frac{8}{5a} \int_0^{a/2} dx \sin^2 \frac{\pi x}{a} \left[ 1 + \cos^2 \frac{\pi x}{a} + 2 \cos \frac{\pi x}{a} \cos \left( \frac{3\pi^2 \hbar t}{2ma^2} \right) \right]$$

$$= \frac{1}{2} + \frac{16}{15\pi} \cos \left( \frac{3\pi^2 \hbar t}{2ma^2} \right)$$

5. Show that $\frac{d}{dt} \int_{-\infty}^{\infty} \psi_1^* \psi_2 dx = 0$ for any two normalizable solutions to the Schrödinger equation, $\psi_1$ and $\psi_2$.

Solution:

$$\frac{d}{dt} \int_{-\infty}^{\infty} \psi_1^* \psi_2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\psi_1^* \psi_2) dx = \int_{-\infty}^{\infty} \left( \frac{\partial \psi_1^*}{\partial t} \psi_2 + \psi_1^* \frac{\partial \psi_2}{\partial t} \right) dx$$

$$= \int_{-\infty}^{\infty} \left( \frac{-i\hbar}{2m} \frac{\partial^2 \psi_1^*}{\partial x^2} - \frac{V}{\hbar} \psi_1^* \right) \psi_2 dx + \psi_1^* \left( \frac{i\hbar}{2m} \frac{\partial^2 \psi_2}{\partial x^2} - \frac{V}{\hbar} \psi_2 \right) dx$$

$$= -\frac{i\hbar}{2m} \int_{-\infty}^{\infty} \left( \frac{\partial^2 \psi_1^*}{\partial x^2} - \frac{\partial^2 \psi_2^*}{\partial x^2} \right) dx$$

$$= -\frac{i\hbar}{2m} \left[ \frac{\partial \psi_1^*}{\partial x} \psi_2 \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \psi_1^*}{\partial x} \frac{\partial \psi_2}{\partial x} dx - \left[ \psi_1^* \frac{\partial \psi_2}{\partial x} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial \psi_1^*}{\partial x} \frac{\partial \psi_2}{\partial x} dx$$

$$= 0$$