Quantum Mechanics I: 1st –Exam Solutions (Fall 2008)

Name______________________________, Panther ID ______________________________

1. To understand atomic energy levels, de Broglie hypothesized that all allowed orbits for an electron in a hydrogen atom are given by the condition that an integral number "n" de Broglie wavelength fit around the circumference of the orbit. Show that this condition leads to the Bohr quantization condition on angular momentum $L = n\hbar$.

Solution:

Let the electron have speed v in an orbit of radius r. The circumference of the orbit is the $2\pi r$. de Broglie's hypothesis is then that an integral number of wave lengths fit into this circumference:

$$\frac{2\pi r}{\lambda} = n.$$

Setting $\lambda = \frac{h}{m_e v}$ gives

$$\frac{2\pi m_e vr}{h} = n.$$

However, the angular momentum of the electron is given by $L = m_e vr$ such that

$$L = \left(\frac{h}{2\pi}\right) n = \hbar n,$$

precisely the Bohr quantization condition of angular momentum.

2. If $\Delta x \Delta p = \hbar/2$ where $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ but $\langle x \rangle = \langle p \rangle = 0$, show that the minimum energy of a quantum harmonic oscillator is $\hbar\omega/2$, if the Hamiltonian is $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$

Solution:

Since $\langle x \rangle = \langle p \rangle = 0$. $\langle \Delta x \rangle^2 = \langle x^2 \rangle$ and $\langle \Delta p \rangle^2 = \langle p^2 \rangle$

$$\langle H \rangle = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} x^2 = \frac{1}{2m}[\langle \Delta p \rangle^2 + m^2\omega^2 \langle \Delta x \rangle^2]$$

For any real value of a and b, $a + b \geq 2\sqrt{ab}$. If $a \equiv \langle \Delta p^2 \rangle$ and $b \equiv m^2\omega^2 \langle \Delta x^2 \rangle$, then

$$\langle \Delta p \rangle^2 + m^2\omega^2 \langle \Delta x \rangle^2 \geq 2\sqrt{\langle \Delta p \rangle^2 \cdot m^2\omega^2 \langle \Delta x \rangle^2} = 2m\omega \langle \Delta x \rangle \langle \Delta p \rangle \geq 2m\omega \frac{\hbar}{2}$$

so that $\langle H \rangle_{\min} = \frac{1}{2}\hbar\omega$

3. Presume that the wave function $\psi(x) = Ae^{-kx}$ applies over the domain $0 \le x \le \infty$ (assume that $\psi = 0$ outside this domain). What must be the value of A in terms of k if this function is to be properly normalized? What is the probability of finding the particle between $x = 0$ and $x = 1/k$?

Solution:

We must have $\int |\psi|^2 dx = 1$. In the present case this demands

$$A^2 \int_0^\infty e^{-2kx} dx = 1$$

such that $A = \sqrt{2k}$

The probability of finding the particle between $x = 0$ and $x = 1/k$ is given by the integral of ψ^2 over these limits

$$P(0,1/k) = \int_0^{1/k} |\psi|^2 dx = 2k\left[-\frac{1}{2k}e^{-2kx}\right]_0^{1/k} = 1 - e^{-2} \sim 0.865$$

4. A particle with mass of m is confined in a one dimensional infinitive well ($0 \le x \le$ a)

i.e. $V(x) = \begin{cases} 0\, when\, 0 \le x \le a \\ \infty otherwise \end{cases}$. And the energy eigenvalues and eigenfunctions are $E_n = \frac{\hbar^2 \pi^2 n^2}{2ma}$ and $\psi_n = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$ with n = 1, 2, 3…, respectively. At t = 0, the initial normalized wave function is $\psi(x,0) = \sqrt{\frac{8}{5a}}(1+\cos\frac{\pi x}{a})\sin\frac{\pi x}{a}$. a) Find the wave function at any moment t; b) What is the expectation value of energy; c) Find the probability in the left half of the well ($0 \le x \le$ a/2) at any moment t.

Solution:

(a) The initial state can be decomposed as the combination of first two eigenstates:

$$\psi(x,0) = \sqrt{\frac{8}{5a}}(1+\cos\frac{\pi x}{a})\sin\frac{\pi x}{a} = \sqrt{\frac{4}{5}}\psi_1(x) + \sqrt{\frac{1}{5}}\psi_2(x)$$

with

$$\psi_1 = \sqrt{\frac{2}{a}}\sin(\frac{\pi x}{a}) \ (n = 1) \text{ and } \psi_2 = \sqrt{\frac{2}{a}}\sin(\frac{2\pi x}{a}) \ (n = 2)$$

According to the superposition principle: $\psi(x,t) = \sum_n C_n \psi_n(x) e^{-\frac{iE_n t}{\hbar}}$, we have

$$\psi(x,t) = \sqrt{\frac{4}{5}}\psi_1(x)\exp(-\frac{i\hbar\pi^2 t}{2ma^2}) + \sqrt{\frac{1}{5}}\psi_2(x)\exp(-\frac{2i\hbar\pi^2 t}{ma^2})$$

$$= \sqrt{\frac{8}{5a}}\sin\frac{\pi x}{a}[\exp(-\frac{i\hbar\pi^2 t}{2ma^2}) + 2\cos\frac{\pi x}{a}\exp(-\frac{2i\hbar\pi^2 t}{ma^2})]$$

(b) Since the energy does not change with time (stationary states), the energy at any moment is given by the initial energy as

$$< H >= E_0 = \frac{4}{5}E_1 + \frac{1}{5}E_2 = \frac{4\hbar^2\pi^2}{5ma^2}$$

(c) The probability at any moment *t* is given by

$$P(t)(0 \le x \le \frac{a}{2}) = \int_0^{a/2} dx \left|\psi(x,t)\right|^2$$

$$= \frac{8}{5a}\int_0^{a/2} dx \sin^2\frac{\pi x}{a}[1 + \cos^2\frac{\pi x}{a} + 2\cos\frac{\pi x}{a}\cos(\frac{3\pi^2\hbar t}{2ma^2})]$$

$$= \frac{1}{2} + \frac{16}{15\pi}\cos(\frac{3\pi^2\hbar t}{2ma^2})$$

5. Show that $\frac{d}{dt}\int_{-\infty}^{\infty}\psi_1^*\psi_2 dx = 0$ for any two normalizable solutions to the Schrödinger equation, ψ_1 and ψ_2.

Solution:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\psi_1^*\psi_2 dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\psi_1^*\psi_2)dx = \int_{-\infty}^{\infty}(\frac{\partial\psi_1^*}{\partial t}\psi_2 + \psi_1^*\frac{\partial\psi_2}{\partial t})dx$$

$$= \int_{-\infty}^{\infty}[(\frac{-i\hbar}{2m}\frac{\partial^2\psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\psi_1^*)\psi_2 + \psi_1^*(\frac{i\hbar}{2m}\frac{\partial^2\psi_2}{\partial x^2} - \frac{i}{\hbar}V\psi_2)]dx$$

$$= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}(\frac{\partial^2\psi_1^*}{\partial x^2}\psi_2 - \psi_1^*\frac{\partial^2\psi_2}{\partial x^2})dx$$

$$= -\frac{i\hbar}{2m}\{\left[\frac{\partial\psi_1^*}{\partial x}\psi_2\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_2}{\partial x}dx - \left[\psi_1^*\frac{\partial\psi_2}{\partial x}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_2}{\partial x}dx\}$$

$$= 0$$