Jensen’s Alpha, Sharpe Ratio, Tracking Error, and Active Portfolio Management

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Portfolio Expected Return and Risk

1. \( X_1, X_2, \ldots, X_N \) portfolio weights on assets 1, 2, ..., \( N \)
2. \( R_i \) rate of return on asset \( i = 1, 2, \ldots, N \)
3. \( \sigma_{ij} = \text{cov}(R_i, R_j) \)
4. \( \sigma_{ii} = \sigma_i^2 = \text{var}(R_i) \)
5. \( E[R_i] \) is the expected return on asset \( i \)
6. expected portfolio return:

\[
E[R_p] = \sum_{i=1}^{N} X_i E[R_i]
\]

7. portfolio variance:

\[
\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}
\]
Sharpe Ratio

1. Definition:

\[ S_P = \frac{E[R_P] - R_f}{\sigma_P} \]

2. Return for portfolio \( Q \) consisting of the \( \theta \) invested in portfolio \( P \) and \( 1 - \theta \) in the risk-free asset:

\[ R_Q = (1 - \theta) R_f + \theta R_P = R_f + \theta (R_P - R_f) \]

3. Expected return and risk for \( Q \):

\[ E[R_Q] = R_f + \theta E[R_P - R_f] \]
\[ \sigma_Q = \theta \sigma_P \]

4. Therefore:

\[ E[R_Q] = R_f + \frac{\sigma_Q}{\sigma_P} (E[R_P] - R_f) = R_f + S_P \sigma_Q \]
1. \[ E[R_Q] = R_f + S_P \sigma_Q \]

2. We can adjust for risk preferences by lending or borrowing.

3. Therefore, the optimal portfolio of risky assets is the one that maximizes the Sharpe ratio.

\[
\begin{array}{c|c|c}
\text{E[R}_Q\text{]} & \text{R}_f & S_P \sigma_Q \\
\hline
\text{Risk} & \text{P}_1 & \text{P}_2 \\
\text{P}_3 & \text{P}_4 \\
\text{Desired Risk} & \text{Q}_1 & \text{Q}_2 \\
\text{Q}_3 & \text{Q}_4 \\
\end{array}
\]
Jensen’s Alpha

1. Form a new portfolio $P^*$ by adding or subtracting small amounts of each security $j = 1, 2, \ldots, N$.

2. Let $Y_j$ be the holding of each security in $P^*$.

3. Then the incremental holding of each security is

$$y_j = Y_j - X_j$$

4. Notice that

$$\sum_{j=1}^{N} y_j = 0$$

5. In other words, the strategy is self-financing: all purchases are financed by sales.

6. The return on portfolio $P^*$ is given by:

$$R_{P^*} = R_P + \sum_{j=1}^{N} y_j R_j$$
Definition for security $i$ with respect to portfolio $P$.

$$\alpha_i = E[R_i] - \beta_i (E[R_P] - R_f)$$

Proposition:

$$\alpha_i > 0 \Leftrightarrow \frac{\partial S_P^*}{\partial y_i} \bigg|_{y_i=0} > 0$$

If Jensen’s alpha is positive for a given security, adding a small amount will increase the Sharpe ratio.

Proposition:

$$\sum_{j=1}^{N} X_j \alpha_j = 0$$

The optimal (or efficient) portfolio has the property that

$$\alpha_i = 0$$

for all securities.
Regression model:

\[ R_{i,t} - R_{f,t} = \alpha_i + \beta (R_{P,t} - R_{f,t}) + \epsilon_{i,t} \]

In terms of excess returns

\[ r_{i,t} = \alpha_i + \beta r_{P,t} + \epsilon_t \]

If portfolio \( P \) is ex post efficient, then

\[ \alpha_i = 0 \]

for all securities \((i = 1, 2, \ldots, N)\) in the portfolio.
We know the average alpha with respect to index $P$ equals zero.

Superior performance requires that we overweight (wrt to the index) securities with positive alphas and underweight those with negative alphas.

If we do this using historical alphas, then our portfolio will have superior historical performance, which does not do us much good.

This is why historical portfolio optimizations always outperform naive diversification strategies.
So we should use forecasted alphas

Since the average of all alphas (wrt to a $P$) is zero, our naive estimate for alpha is zero for any security contained within $P$

We should follow an active strategy only if we believe we can successfully forecast alpha

That means our forecasted alphas on average are significantly different from zero on a subset of securities
1. Determine universe of securities
2. Divide universe into asset classes
3. Desirable qualities
   1. investable
   2. can be matched to an index
   3. correlations between asset classes relatively low
   4. correlation between securities within asset class relatively high
Asset Allocation Diagram

Universe of Securities

Asset Class 1 ($l_1$)
Asset Class 2 ($l_2$)
Asset Class N ($l_N$)
1. Let \( w_1, w_2, \ldots, w_N \) be the weights in each asset class.

2. Then overall passive portfolio return

\[
R_P = W_1 l_1 + W_2 l_2 + \ldots + W_N l_N
\]

3. Active strategy will differ because

   1. different asset class weighting
   2. different weighting within each class
Active (managed) return:

\[ R_A = X_1 R_1 + X_2 R_2 + \ldots + X_N R_N \]

Return attribution:

1. asset allocation

\[(X_1 - W_1) I_1 + (X_2 - W_2) I_2 + \ldots + (X_N - W_N) I_N\]

2. security selection

\[(R_1 - I_1) X_1 + (R_2 - I_2) X_2 + \ldots + (R_N - I_N) X_N\]

3. total

\[ R_A - R_P = \text{asset allocation} + \text{security selection} \]

\[ = (R_1 X_1 - W_1 I_1) + \ldots + (R_N X_N - W_N I_N) \]
Asset Allocation Methods

1. Two-step method
   1. choose asset weights
   2. choose portfolios in each asset class

2. Three-step method
   1. choose asset weights
   2. hire asset-class managers
   3. managers choose portfolios in each class
Three-Step Method with Passive Asset Allocation

1. Passive:
   1. set strategic policy weights (say using portfolio optimization)
   2. choose index funds matched to asset classes
   3. evaluate performance of asset allocation

2. Active:
   1. set strategic policy weights
   2. set maximum tracking error
   3. choose portfolio managers
   4. evaluate performance of each fund manager
   5. evaluate performance of asset allocation
Tracking Error

1. Definition for asset class $i$:

$$TE_i = \sqrt{\text{var}(R_i - l_i)}$$

2. Properties:

$$\text{var}(R - l) = \text{var}(R) + \text{var}(l) - 2 \text{cov}(R, l)$$

3. The active return is traded-off against the active risk (measured by the tracking error).

4. Reasons for tracking error constraint:
   - ensure that active portfolio matches index to some extent
   - limit exposure to detrimental active management