Bond Price Volatility

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example: $CR = 6\%$, $T = 20$ years, $M = $100

value: $P(y) = 3A(y/2, 40) + \frac{100}{(1+y/2)^{40}}$

declining

convex
Bond price properties

1. Prices are inversely related to yield.
2. For a small yield change (either up or down) the price sensitivity can be measured by the slope.
3. For a large change in yield the price sensitivity is greater when yields decrease.
4. Sensitivity is negatively related to the yield.
Characteristics of a bond that explain price volatility

1. positively related to maturity
2. price versus yield for coupon rate of 6% with 5, 15, and 30 year maturities
Characteristics of a bond that explain price volatility

1. negatively related to coupon rate
2. value per dollar invested (assuming yield initial yield of 6 percent) versus yield for coupon rate of 0, 6, and 12 percent
Duration analysis

1. modified duration

\[ MD \equiv \frac{1}{P} \left| \frac{dP}{dy} \right|_{y=y_0} \approx \frac{\Delta\%P}{\Delta y} \]

for small \( \Delta y \)

2. Macaully Duration:

\[ D \equiv \frac{1 + y}{P} \left| \frac{dP}{dy} \right| \]

3. formula:

\[ D = \frac{1}{P} \left( \frac{C_1}{(1+y)} + \frac{2C_2}{(1+y)^2} + \cdots + \frac{NC_N}{(1+y)^N} \right) \]

4. for semi-annual bonds substitute \( \frac{y}{2} \) for \( y \) and multiply \( D \) by \( \frac{1}{2} \)

5. relationship

\[ D = (1 + y) \cdot MD \]
1. Define present value cash flow weight for cash flow promised at date \( i \):

\[
w_i = \frac{C_i / (1 + y)^i}{P}
\]

2. Weights sum to one

3. Then

\[ D = w_1 + w_2 2 + \ldots + w_N N \]

4. So duration is a weighted average of cash flow payment dates
Properties of Macaully duration

1. measures price sensitivity to interest rate change:

\[ \Delta \% P \approx - \left( \frac{D}{1 + y} \right) \Delta y \]

2. in units of time (years)

3. measures effective life of bond

4. duration of zero equals maturity \((w_N = 1)\)

5. duration of coupon bond less than zero with same maturity

6. duration increases in maturity

7. for a coupon bond duration increases with maturity at a decreasing rate

8. duration decreases with \(CR\)
Duration increases with maturity

- $CR = 0, 2, 6, \text{ and } 10\%$:
Duration decreases with yield

- 5, 15, and 30 year maturity bond with CR = 6%
**Duration analysis**

**Example**

- $CR = 6\%$, $T = 20$ years, $M = \$100$

1. if $y = 6\%$, then

   \[
   P = \$100 \text{ and } D = 11.904 \text{ years}
   \]

2. suppose $\Delta = +50$ bps

3. $\Delta\%P \approx -\frac{11.904}{1.03}(.5) = -5.779\%$

4. suppose $\Delta = -50$ bps

5. $\Delta\%P \approx -\frac{11.904}{1.03}(-.5) = +5.779\%$
Duration analysis

Example – continued

1. approximate new price at $y = 5.5\%$

$$100(1 + .05779) = 105.78$$

2. actual price at

$$P = 3A(.055/2, 40) + \frac{100}{(1 + .055/2)^{40}}$$

$$= 106.02$$

3. approximate new price at

$$100(1 - .05779) = 94.221$$

4. actual price at $y = 6.5\%$

$$P = 3A(.065/2, 40) + \frac{100}{(1 + .065/2)^{40}}$$

$$= 94.448$$
Convexity

1. actual price always greater than duration approximation
2. why?
3. because price-yield curve is convex, linear approximation at initial yield is a supporting line

![Graph showing the relationship between price and yield, illustrating the convexity concept.](Image)
Convexity

1. Convexity adjustment:

\[ \Delta \%P \approx -\left( \frac{D}{1+y} \right) \Delta y + \frac{1}{2} CX (\Delta y)^2 \]

where \( CX \) measures convexity at \( y \)

2. Note convexity adjustment is always positive

3. Definition:

\[ CX = \frac{1}{P} \frac{d^2 P}{dy^2} \]

4. Formula (do not need to know):

\[ CX = \frac{1}{(1+y)^2} \left( \sum_{n=1}^{N} w_n \left( \frac{n}{2} \right)^2 + D \right) \]
Convexity measures cash flow dispersion about duration

1. bullet (zero coupon)
2. ladder (annuity)
3. barbell (use your imagination)
Example 1

- compare $CR = 6\%$, $T = 20$ years, $M = $100
- to zero with $T = 11.904$ years and $P = $100

Both bonds have the same duration and the same price.

Need to solve for $M$:

$$100 = \frac{M}{(1.03)^{11.904 \times 2}}$$

Solution is: $\{M = 202.13\}$

So

$$P_{ZC}(y) = \frac{202.13}{(1 + y/2)^{11.904 \times 2}}$$
Convexity measures cash flow dispersion about duration

Example 1: price yield curve for coupon bond and zero
Example 2

- compare bullet, barbell, and ladder
- same price and duration
- term structure: flat 6%

1. ladder is 20 year $6 annuity, semiannual payment
2. price of ladder:
   \[ 3A(0.03, 40) = 69.344 \]
3. duration of ladder equals 8.325 years
4. zero coupon bond promises 113.440 dollars in 8.325 years
5. price of zero coupon:
   \[ P = \frac{113.440}{(1 + 0.03)^{-2 \times 8.325}} = $69.344 \]
6. duration by construction equals 8.325 years
barbell makes two payments at 2 and 20 years of 50.622 and 79.485 dollars

price

\[
\frac{50.622}{(1.03)^4} + \frac{79.485}{(1.03)^{40}} = 69.344
\]

duration

\[
D = \left( \frac{50.622 \times 2}{(1.03)^4} + \frac{79.485 \times 20}{(1.03)^{40}} \right) \div 69.344 \\
= 8.325
\]
Convexity measures cash flow dispersion about duration

Example 2: price-yield curve for zero, ladder, and barbell
Convexity measures cash flow dispersion about duration

Example 2: price minus linear approximation
Effective (approximate) duration and convexity

1. approach for dealing with bonds with embedded options
2. based upon valuation model estimate the slope of price-yield curve about market yield
3. currently observed market price is $P_0$
4. compute the price $P^+$ for a small yield increase $y + \Delta y$
5. compute the price $P^-$ for a small yield decrease $y - \Delta y$
6. define slope
   \[ \text{slope} = \frac{P^+ - P^-}{2\Delta y} \]
7. approximate duration
   \[ \frac{1}{P_0} \frac{dP}{dy} \approx \frac{1}{P_0} |\text{slope}| = \frac{P^+ - P^-}{2P_0\Delta y} \]
Effective (approximate) duration and convexity – continued

approximate convexity

\[
\frac{1}{P_0} \frac{d^2 P}{dy^2} \approx \frac{1}{P_0} \left[ \frac{(P_+ - P_0)}{\Delta y} - \frac{(P_0 - P_-)}{\Delta y} \right] / \Delta y
\]

\[
= \frac{(P_+ + P_- - 2P_0)}{P_0 (\Delta y)^2}
\]
Effective (approximate) duration and convexity

Example

1. \( P_0 = 90 \) and \( y = 6\% \)
2. 25 bps shock: \( y^+ = 6.25\% \) ⇒ \( P(y^+) = 88 \)
3. up percentage change = \( (88 - 90)/90 = -\frac{1}{45} = -2.22\% \)
4. \( y^- = 5.75\% \) ⇒ \( P(y^-) = 92.75 \)
5. down percentage change = \( (92.7 - 90)/90 = 3.00\% \)
6. approximate duration

\[
\frac{92.7 - 88}{90 \times .005} = 10.44
\]

7. approximate convexity

\[
\frac{92.7 + 88 - 180}{90 \times .0025^2} = 1244.4
\]
Yield curve risk

1. above analysis assumed parallel shift
2. risk of change in the shape of yield curve
3. rate duration - sensitivity of bond to change in given spot rate
4. set (or vector) key rate durations
5. overall duration is weighted sum of rate durations
Yield curve risk

Example

- barbell that promises $100 and $200 in 3 and 10 years
- flat term 6% term structure

1. suppose 3 year rate decreases by 50 bps
2. and 10 year rate increases by 50 bps
3. steepening of yield curve
4. initial price:

\[
\frac{100}{1.03^6} + \frac{200}{1.03^{20}} = 83.748 + 110.74 = 194.48
\]

5. duration:

\[
\frac{83.748 \times 3 + 110.74 \times 10}{194.48} = 6.986
\]
Yield curve risk
– continued

1 key rate durations

3 and 10 years

2 based upon rate durations

\[ \Delta P = 83.748 \left( \frac{3}{1.03} \right) (0.005) - 110.74 \left( \frac{10}{1.03} \right) (0.005) \]

\[ = -4.1561 \]

3 or a new price of

\[ 194.48 - 4.1561 = 190.32 \]

4 actual new price:

\[ \frac{100}{(1 + 0.055/2)^6} + \frac{200}{(1 + 0.065/2)^{20}} \]

\[ = 190.47 \]
Key rate durations

1. each payment date has a rate duration
2. idea: pick out a small set of key maturities
3. for example: 3 months, 1, 2, 3, 5, 7, 10, 15, 20, 25, 20 years
4. key rates are spot rates with key maturities
5. key rate duration is sensitivity of a bond portfolio to a given change in a key rate
Multifactor duration

1. based upon historical term structure changes, identify key term structure shifts
2. define a duration measure with respect to each key shift
3. advantages
4. considers interrelationships between spot rate changes
5. fewer key durations (2 or 3 rather than 11)
6. gives guidance as to what type of shifts are likely to occur
7. and what types of shifts you are protected against