CAPM and Market Model

• If market portfolio is mean-variance efficient, then
  \[ E[R_i] = R_F + \beta(E[R_m] - R_F) \]

• Market Model
  
  \[ R_{it} = \alpha_i + \beta_i R_m + \varepsilon_i \]

  – Choose \( \alpha \) and \( \beta \) such that the sample variance of the residual \( \varepsilon \) is minimum:
    \[ \beta_i = \frac{\hat{\text{cov}}(R_i, R_m)}{\hat{\text{var}}(R_m)} \]
    and
    \[ \alpha_i = \overline{R}_i - \beta_i \overline{R}_m \]

  – Estimators also have the property that the sample correlation between the residual and market return equals zero.
– Summary of properties

* Unsystematic risk is minimum

* Average residual is zero so the average stock and market return fall on regression line

* Sample correlation between residual and market return equals zero

– Residual across stocks may be correlated, suggesting that additional factors may be required
• CAPM and Market Model

  – Assume market model beta and CAPM beta are the same.

  – Then the CAPM places restriction on the value of $\alpha$.

  – Under market model:
    \[ E[R] = \alpha + \beta E[R_m] \]

  – Therefore,
    \[ \alpha = R_F (1 - \beta) \]

  – If $\beta > 1$, $\alpha < 0$

  – If $\beta < 1$, $\alpha > 0$
Example

- Suppose $E[R_m] = 10\%$ and $R_F = 6\%$.

- Consider four stocks with betas of 0, .5, 1, and 1.5.

- If CAPM holds, then alphas are given by 6\%, 3\%, 0, −3\%.

- In up market, high beta stocks tend to perform better.
- For example, suppose $R_m = 20\%$

- In down market, low beta stocks tend to perform better
  - For example, $R_m = -5\%$
- But over up and down markets, high beta stocks have higher average returns
• Flat relationship: alphas given by 10%, 5%, 0%, -5%

• Given that for large portfolios,

\[
\sigma^2_p = \beta_p^2 \sigma^2_m + \sum_{j=1}^{N} \left( \frac{1}{N} \right)^2 \text{var}(\varepsilon_i) \\
+ \sum_{j \neq k} \left( \frac{1}{N} \right)^2 \text{cov}(\varepsilon_j, \varepsilon_k) \\
= \beta_p^2 \sigma^2_m + \frac{\text{var}(\varepsilon_i)}{N} +
\]
\[
\left( \frac{1}{N} \right)^2 \left( N^2 - N \right) \text{cov}(\varepsilon_j, \varepsilon_k) = \beta_p^2 \sigma_m^2 + \frac{\text{var}(\varepsilon_i)}{N} + \text{cov}(\varepsilon_j, \varepsilon_k) \left( 1 - \frac{1}{N} \right)
\]

\[ \lim_{N \to \infty} \sigma_p^2 = \beta_p^2 \sigma_m^2 + \text{cov}(\varepsilon_j, \varepsilon_k) \]

- a portfolio of low beta stocks will have lower total risk.

- For a flat relationship, risk averse investors will choose low-beta risky portfolio and adjust risk by borrowing or lending at risk-free rate.

- If expected return on a portfolio is greater than risk-free rate, one can always increase the expected return and beta with leverage.

- Suppose \( \beta_U \) is beta of unlevered portfolio, then beta of levered portfolio is

\[ \beta_L = X \beta_U + (1 - X)(0) \]
– Expected return is given by

\[ E[R_{LEV}] = R_F + X(E[R_m] - R_F) \]

– So

\[ E[R_{LEV}] = R_F + \frac{\beta_L}{\beta_U}(E[R_m] - R_F) \]

– If unlevered portfolio is the market,

\[ E[R_{LEV}] = R_F + \beta_L(E[R_m] - R_F) \]

– Also,

\[ \beta_L = X = \frac{\sigma_{LEV}}{\sigma_m} \]