1. There are two goods. Suppose a consumer has the convex utility function \( u(x_1, x_2) = x_1 + x_2^2 \) and consumption set \( \mathcal{X} = \mathbb{R}_+^2 \). Let prices be \( p = (1, p) \gg 0 \) and income be \( m > 0 \). Suppose further that \( \bar{u} \geq 0 \). If possible for this utility function:
   a) Find the Marshallian (ordinary) demand \( x(p, m) \).
      \textbf{Answer:} The indifference curves are parabolas. Because of the convexity of the utility function, maximum utility will be at a corner. The two possibilities are \((m, 0)\) and \((0, m/p)\). If \( p^2 > m \), the first one is better, if \( p^2 < m \), the second is better, and if \( p^2 = m \), they are tied. It follows that the Marshallian demand is:
      \[
      x(p, m) = \begin{cases} 
      (m, 0) & \text{if } p^2 > m \\
      \left\{ (m, 0), (0, \frac{m^2}{p^2}) \right\} & \text{if } p^2 = m \\
      \left( 0, \frac{m}{p} \right) & \text{if } p^2 < m
      \end{cases}
      \]
   b) Compute the indirect utility function \( v(p, m) \).
      \textbf{Answer:} Indirect utility is \( v(p, m) = u(x(p, m)) = \max \{ m, m^2/p^2 \} \).
   c) Find the Hicksian (compensated) demand \( h(p, \bar{u}) \).
      \textbf{Answer:} The convex indifference curves imply that expenditure minization also occurs at a corner. We set \( u(x) = \bar{u} \) to find the corners are \((\bar{u}, 0)\) and \((0, \sqrt{\bar{u}})\). The associated expenditures are \( \bar{u} \) and \( p\sqrt{\bar{u}} \). We choose the first if \( \sqrt{\bar{u}} < p \) and the second if \( \sqrt{\bar{u}} > p \). Thus
      \[
      h(p, \bar{u}) = \begin{cases} 
      (\bar{u}, 0) & \text{if } \sqrt{\bar{u}} < p \\
      \left\{ (\bar{u}, 0), (0, \sqrt{\bar{u}}) \right\} & \text{if } \sqrt{\bar{u}} = p \\
      (0, \sqrt{\bar{u}}) & \text{if } \sqrt{\bar{u}} > p
      \end{cases}
      \]
   d) Compute the expenditure function \( e(p, \bar{u}) \).
      \textbf{Answer:} Expenditure is \( e(p, \bar{u}) = p \cdot h(p, \bar{u}) = \min \{ \bar{u}, p\sqrt{\bar{u}} \} \).
2. There are two consumers and two goods. Consumer 1 has utility \( u_1(x) = (x_1x_2)^{1/2} \) and income \( m_1 > 0 \). Consumer 2 has utility \( u_2(x) = \min(x_1, 2x_2) \) and income \( m_2 > 0 \). Prices are \( p = (p_1, p_2) \gg 0 \).
   a) Find the Marshallian demand functions for each consumer.
      \textbf{Answer:} Equal-weighted Cobb-Douglas utility requires consumer 1 spend half of his income on each good, so \( x^1(p, m_1) = (m_1/2)(1/p_1, 1/p_2) \). The second consumer has Leontief preferences with \( x_1 = 2x_2 \) at the optimum. Total spending is then \( m_2 = p_1x_1 + p_2x_2 = (2p_1 + p_2)x_2 \) so \( x^2(p, m_2) = (2m_2/(2p_1 + p_2), m_2/(2p_1 + p_2)) \).
   b) Compute aggregate demand.
      \textbf{Answer:} We add the consumer demands to obtain aggregate demand. It is
      \[
      x(p, m_1, m_2) = \left( \frac{m_1}{2p_1} + \frac{2m_2}{2p_1 + p_2}, \frac{m_1}{2p_2} + \frac{m_2}{2p_1 + p_2} \right).
      \]
   c) Show by example that aggregate demand cannot be written as a function of the price vector \( p \) and aggregate wealth \( m = m_1 + m_2 \).
      \textbf{Answer:} Now \( x(p, 2, 0) = (1/p_1, 1/p_2) \) and \( x(p, 0, 2) = (2p_1 + p_2)^{-1}(4, 2) \). Since these only agree when \( 2p_1 = p_2 \), the different income distributions yield different demand curves.
3. Suppose $Y$ is a convex technology set (also is non-empty, closed, obeys inaction, no free lunch, and free disposal). Suppose $y \gg 0$. Show there is a price vector with $p \cdot y > \pi(p)$ (20 points). Show that $p \geq 0$ (5 points).

**Answer:** Now $Y$ is a closed convex set and $y$ is a point outside the set. By the Separation Theorem, there are $p \neq 0$ and $\alpha \in \mathbb{R}$ with $p \cdot y > \alpha > p \cdot z$ for all $z \in Y$. Taking the supremum over $z \in Y$, we find $y \cdot y > \alpha > \pi(p)$.

Now consider $-n\ell \in Y$ by free disposal for $n > 0$. Then $\alpha > p \cdot (-n\ell) = -np\ell$. It follows that $\alpha/n > -p\ell$. Letting $n \to +\infty$, we find $0 \leq p\ell$. Since $\ell$ was arbitrary, and $p \neq 0$, $p > 0$.

4. A consumer with $X = \mathbb{R}^2_+$ has expenditure function $e(p, \bar{u}) = p_1 + p_2 + \sqrt{p_1 p_2} + (p_1 + 2p_2)\bar{u}$.

a) Find the Hicksian demand function $h(p, \bar{u})$.

**Answer:** Note that $e$ is concave and homogeneous of degree 1 in prices, as an expenditure function should be. We apply the Shephard-McKenzie Lemma, which says $h = D_p e$. Then

$$h(p, \bar{u}) = \left( 1 + \frac{1}{2} \sqrt{\frac{p_2}{p_1}} + \bar{u}, 1 + \frac{1}{2} \sqrt{\frac{p_1}{p_2}} + 2\bar{u} \right)$$

b) Find the indirect utility function $v(p, m)$.

**Answer:** We use the duality relation $m = e(p, v(p, m))$ to find indirect utility. Then $m = p_1 + p_2 + \sqrt{p_1 p_2} + (p_1 + 2p_2)v(p, m)$, so

$$v(p, m) = \frac{m - p_1 - p_2 - \sqrt{p_1 p_2}}{p_1 + 2p_2}$$

c) Find the Marshallian demand function $x(p, m)$.

**Answer:** According to Roy’s Identity, $x_\ell(p, m) = -(\partial v/\partial p_\ell)/(\partial v/\partial m)$. Now $\partial v/\partial m = 1/(p_1 + 2p_2)$. After a short calculation, we find

$$x_1(p, m) = 1 + \frac{1}{2} \sqrt{\frac{p_2}{p_1}} + \frac{m - p_1 - p_2 - \sqrt{p_1 p_2}}{p_1 + 2p_2} = \frac{m + p_2 - \frac{1}{2} \sqrt{p_1 p_2} + p_2 \sqrt{\frac{p_2}{p_1}}}{p_1 + 2p_2}$$

$$x_2(p, m) = 1 + \frac{1}{2} \sqrt{\frac{p_1}{p_2}} + \frac{2m - p_1 - p_2 - \sqrt{p_1 p_2}}{p_1 + 2p_2} = \frac{2m - p_1 - \sqrt{p_1 p_2} + \frac{p_1}{2} \sqrt{\frac{p_1}{p_2}}}{p_1 + 2p_2}$$