Homework #8

12.1 Suppose \( u_1(x^1) = (x_1^1)^{1/3}(x_2^1)^{2/3} \) and \( u_2(x^2) = (x_1^2)^{1/3}(x_2^2)^{2/3} \), with endowments \( \omega^1 = (7, 1) \) and \( \omega^2 = (3, 1) \). Find the core.

**Answer:** Since the consumers have identical Cobb-Douglas preferences, the Pareto set is the diagonal of the Edgeworth box. The aggregate endowment is \( \omega = (10, 2) \) and total utility is \( 10^{1/3}4^{2/3} = 160^{1/3} \). Individual rationality requires \( u_1 \geq u_1(\omega^1) = 7^{1/3} \) and \( u_2 \geq u_2(\omega^2) = 3^{1/3} \). Thus the core is \( C(E) = \{ (u_1(10, 2), (1 - u_1((10, 2))) : u_1 \geq 7^{1/3}, u_2 \geq 3^{1/3}, u_1 + u_2 = 160^{1/3} \} \)

12.2 Suppose \( u_1(x^1) = \min\{x_1^1, \beta x_2^1\} \) and \( u_2(x^2) = \min\{x_1^2, x_2^2\} \), with endowments \( \omega^1 = (1, 1) \) and \( \omega^2 = (\alpha, 1) \). Find the core.

**Answer:** The (a) diagrams illustrate the case where \( \alpha > 1 \) while the (b) diagrams have \( \alpha < 1 \). The dashed lines are the corner points for the Leontief preferences. As far as the Pareto set is concerned, the main issue is whether or not these lines intersect. The non-intersection case (a1, b1) is on the left, and the intersection case in on the right (a2, b2). The Pareto set is hatched, with the hatch lines running along the mutual indifference curves.

Core points are both Pareto optimal and individually rational. The heavy dot is the endowment, and the red lines denote the indifference curves through the endowment. If the endowment is Pareto optimal, the core follows the hatch lines as in (b1). If the endowment is not Pareto optimal, the core consists of the portion of the Pareto set within the red box.

12.4 Suppose \( u_1(x^1) = \max\{x_1^1, x_2^1\} \) and \( u_2(x^2) = \min\{x_1^2, x_2^2\} \), with endowments \( \omega^1 = (1, 1) \) and \( \omega^2 = (3, 0) \). Find the core. Be careful, \( u_1 \) is not a typo!

**Answer:** We start by finding the Pareto set. Note that \( u_2 \leq \omega_2 = 1 \). If \( x_1^2 > x_2^2 \), we can Pareto improve by giving the excess of good 1 to consumer 1 who will then have \( u_1 = 4 - x_2^2 \). Thus \( x_1^2 \leq x_2^2 \) at any
Pareto optimum. Once we have given at least $4 - x_2^2$ of good 1 to consumer 1, consumer 1’s utility cannot be further increased. It follows that the Pareto optima are $\{(4 - x, 1 - y), (x, y) : 0 \leq y \leq 1, x \leq y\}$ yielding utility $u_1 = 4 - x$ and $u_2 = x$. (This is the same as problem 11.4.)

Individual rationality requires $u_1 = 4 - x \geq 1$ and $u_2 \geq 0$, both of which are satisfied at all the Pareto optima.

12.8 Suppose utility is $u_i(x) = \sqrt{x_1^i x_2^i}$ for $i = 1, 2, 3$ and endowments are $\omega^1 = (1, 2), \omega^2 = (1, 3)$, and $\omega^3 = (4, 1)$. Find the core.

**Answer:** Again we have identical Cobb-Douglas utility, and the Pareto set (in utility space) is $\{(u_1, u_2, u_3) \in \mathbb{R}^3_+ : u_1 + u_2 + u_3 = 6\}$. We additionally have to satisfy individual rationality: $u_1 \geq \sqrt{2}, u_2 \geq \sqrt{3}$ and $u_3 \geq 2$. We also must be at least as well off as in the Pareto optima for 2-consumer coalitions. Thus $u_1 + u_2 \geq \sqrt{10}, u_1 + u_3 \geq \sqrt{15}$ and $u_2 + u_3 \geq \sqrt{20}$.

We can simplify the conditions by substituting $u_3 = 6 - u_1 - u_2$. Then we obtain: $\sqrt{2} \leq u_1 \leq 6 - \sqrt{20}$, $\sqrt{3} \leq u_2 \leq 6 - \sqrt{15}$, and $\sqrt{10} \leq u_1 + u_2 \leq 4$ together with $u_3 = 6 - u_1 - u_2$.

Any $u_i \geq 0$ that meet the above conditions, such as $(1\frac{1}{2}, 2, 2\frac{1}{2})$ are core utility allocations. The corresponding goods allocations are $x^i = u_i(1, 1)$. 