Introduction to Binomial Trees

Chapter 12
A Simple Binomial Model

- A stock price is currently $20.
- In three months it will be either $22 or $18.

\[
\text{Stock price} = 20 \\
\text{Stock Price} = 22 \\
\text{Stock Price} = 18
\]
A Call Option  (Figure 12.1, page 274)

A 3-month call option on the stock has a strike price of 21.

Stock price = $20  
Option Price=?

Up Move  
Stock Price = $22  
Option Price = $1

Down Move  
Stock Price = $18  
Option Price = $0
Setting Up a Riskless Portfolio

- For a portfolio that is long $\Delta$ shares and short 1 call option, the terminal values are:

$$\Delta S_0 - C$$

- Portfolio is riskless when

$$22\Delta - 1 = 18\Delta$$

or when $\Delta = 0.25$
Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:
  - long 0.25 shares
  - short 1 call option

- The value of the portfolio in 3 months is:
  \[ 22 \times 0.25 - 1 = 4.50 \]
  or \[ 18 \times 0.25 = 4.50 \]

- The value of the portfolio today is:
  \[ 4.5e^{-0.12 \times 0.25} = 4.3670 \]
Valuing the Option

- The portfolio that is:
  - long 0.25 shares
  - short 1 call option

  is worth today $4.367 = \Delta S_0 - C$

- The value of the $\Delta$ shares today is:

  $\Delta S_0 = 0.25 \times 20 = 5.000$

- The value $C$ of the call option today is therefore:

  $C = \Delta S_0 - 4.367 = 5.000 - 4.367 = 0.633$
A Put Option

A 3-month put option on the stock has a strike price of 21.

Stock price = $20  
Option Price = ?

Up Move

Stock Price = $22  
Option Price = $0

Down Move

Stock Price = $18  
Option Price = $3
Setting Up a Riskless Portfolio:

- For a portfolio that is long $\Delta$ shares and short 1 put option, the terminal values are:

  \[ \Delta S_0 - P \]

  - Up Move: $22\Delta$
  - Down Move: $18\Delta - 3$

- Portfolio is riskless when
  \[ 22\Delta = 18\Delta - 3 \quad \text{or when} \quad \Delta = -0.75 \]
Valuing the Portfolio \((R_f = 12\%)\)

- The riskless portfolio is:
  - long –0.75 shares (i.e. short 0.75 shares)
  - short 1 put option

- The value of the portfolio in 3 months is:
  - \(22 \times (-0.75) = -16.50\)
  - Or \(18 \times (-0.75) - 3 = -16.50\)

- Since riskless, the value of the portfolio today is:
  \[-16.50 e^{-0.12 \times 0.25} = -16.012\]
Valuing the Option

- The portfolio that is:
  - long -0.75 shares (i.e. short 0.75 shares)
  - short 1 put option

  is worth today \(-16.012 = \Delta S_0 - P\)

- The value of the \(\Delta\) shares today is:
  \[\Delta S_0 = -0.75 \times 20 = -15.000\]

- The value \(P\) of the put option today is therefore:
  \[P = \Delta S_0 + 16.012 = -15.000 + 16.012\]
  \[P = 1.012\]
Remarks #1

- Note that for the call option, the riskless portfolio is constructed by buying $\Delta$ shares and selling/writing the call option.
- Since $\Delta$ is always positive for a call option, buying $\Delta$ shares is the same as buying $|\Delta|$ shares.

- For the put option, the riskless portfolio is constructed by buying $\Delta$ shares and selling/writing the put option.
- Since $\Delta$ is always negative for a put option, buying $\Delta$ shares is the same as shorting $|\Delta|$ shares.
Remarks #2

- Note that $\Delta$ does not have to be solved for each time as the “plug-in” figure that makes the portfolio riskless.
- It will always be the ratio of the difference in the option prices to the difference in the stock prices in the next period (with a small adjustment if dividends occur).

- In other words:
- $\Delta = (C_u - C_d)/(S_u - S_d)$ for the call
- $\Delta = (P_u - P_d)/(S_u - S_d)$ for the put

- $\Delta$ reflects the slope of the option pricing function.
Remarks #3

- $\Delta$ represents by how much the option changes in value given a change in the price of the underlying asset.
- If the stock price changes by one dollar, the option price approximately changes by $\Delta$ dollars.
- Example: $\Delta=0.30$ for a call option and $\Delta=-0.25$ for a put.
  - If the stock goes up by $1$, the call appreciates by $0.30$ (30 cents)
  - If the stock goes up by $1$, the put depreciates by $0.25$ (25 cents)

- Therefore, if instead of owning one share of the stock, you only own a fraction of a share, $\Delta$ share, the dollar movement of that $\Delta$ share matches that of the option.
- Being long one and short the other creates a riskless asset.
Also note that in order to create the riskless portfolio, all you need is to be long on one side and short on the other.

Therefore, for the call option, a riskless portfolio could also have been created by buying the call and shorting $\Delta$ shares, i.e. by composing the $C-\Delta S$ portfolio.

For the put option, a riskless portfolio could also have been created by buying the put and shorting $\Delta$ shares, i.e. by composing the $P-\Delta S$ portfolio.

But recall that for the put, $\Delta$ is negative. The portfolio can thus also be expressed as $P+|\Delta|S$.

In plain English, you “buy the put and short a negative number of shares”, i.e. you “buy the put and buy a positive number of shares”.

One can also check graphically that the positions being added ($C$, $P$, $\Delta S$, $-\Delta S$, or $|\Delta|S$) have opposite slopes: thus they offset each other.
Remarks #5

- Recall that in the case of the call option, we managed to replicate a risk-free payment, as if a risk-free investment B were returned with interest to you at expiration, hence giving you a payoff of $B e^{rT}$.
- Therefore we have: $\Delta S_T - C_T = B e^{rT}$ and so $\Delta S_0 - C = B$.
- Thus we have today: $C = \Delta S_0 - B$  
  (and $C_T = \Delta S_T - B e^{rT}$)
- This means that we can replicate the Call option future payoffs by combining an investment of $\Delta$ shares in the stock today and shorting a risk-free investment (at time 0, the amount B shorted/borrowed is the present value of the risk-free payment made at expiration).
- The call option price today is thus the cost of this strategy.
- In our previous example, $\Delta = 0.25$, $S_0 = 20$, and $B = 4.5e^{-0.12 \times 0.25}$
- Consequently the call option price is $C = (0.25)(20) - 4.5e^{-0.12 \times 0.25}$
- And so the call option price is $C = 0.633$ (same value as before)
Remarks #6

- Recall that in the case of the put option, we also managed to replicate a risk-free payment, as if a risk-free loan \( B \) were to be paid back by you at expiration with interest, hence costing you a payment of \( B e^{rT} \).
- Therefore we have: \( \Delta S_T - P_T = -B e^{rT} \) and so \( \Delta S_0 - P = -B \).
- Thus we have today: \( P = \Delta S_0 + B \) (and \( P_T = \Delta S_T + B e^{rT} \))
- This means that we can replicate the Put option future payoffs by combining a shorting of \(|\Delta|\) shares in the stock today and investing in a risk-free bond (at time 0, the amount \( B \) bought/invested is the present value of the risk-free payment received at expiration).
- The put option price today is thus the cost of this strategy.
- In our previous example, \( \Delta = -0.75 \), \( S_0 = 20 \), and \( B = 16.5e^{-0.12 \times 0.25} \)
- Consequently the put option price is \( P = (-0.75)(20) + 16.5e^{-0.12 \times 0.25} \)
- And so the put option price is \( P = 1.012 \) (same value as before)
The current stock price is $40, the riskless rate is 8%, and in one year the stock will either be worth $50 or $30.

There is both a call option and a put option on that stock, with a strike price K of $40.

What is the price of the call?
What is the price of the put?

In each case, how much would you need to borrow or invest in order to replicate the call/put option payoffs? (while having an investment of delta shares of the stock)
Additional Example: answers

- Call option price: $C = 6.153$
- Amount borrowed to replicate the Call: $13.847$
- Put option price: $P = 3.078$
- Amount invested to replicate the Put: $23.078$
Graphical Example

Option Payoff

$C_u = uS - K$

$C_d = 0$

Intercept = $e^{rhB}$

Slope = $\Delta$

Rise = $C_u - C_d$

$dS$ $K$ $uS$

$S_h$ (Stock price after one period)

Run = $uS - dS$
A derivative lasts for a time $T$ and its value is dependent on the price of a stock.

\[ S \xrightarrow{\text{Up Move}} Su \]
\[ S \xrightarrow{\text{Down Move}} Sd \]
\[ f \xrightarrow{\text{Up Move}} fu \]
\[ f \xrightarrow{\text{Down Move}} fd \]
**Generalization (continued)**

- Value of a portfolio that is long $\Delta$ shares and short 1 derivative:

\[
\begin{align*}
\text{Up Move} & : S_0 u \Delta - f_u \\
\text{Down Move} & : S_0 d \Delta - f_d
\end{align*}
\]

- The portfolio is riskless when $S_0 u \Delta - f_u = S_0 d \Delta - f_d$

or, by rearranging, when: $\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$
The value of the portfolio at time $T$ is: $S_u \Delta - f_u$

The value of the portfolio today is: $(S_u \Delta - f_u) e^{-rT}$

Another expression for the portfolio value today is: $S \Delta - f$

Hence we have: $f = S \Delta - (S_u \Delta - f_u) e^{-rT}$
Generalization (continued)

But $\Delta$ can be expressed as: 

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Therefore, substituting $\Delta$, we obtain:

$$f = [pf_u + (1 - p)f_d]e^{-rT}$$

where 

$$p = \frac{e^{rT} - d}{u - d}$$
\textbf{$p$ as a Probability}

- It is natural to interpret $p$ and $1-p$ as the probabilities of up and down movements.

- The value of a derivative is then its expected payoff discounted at the risk-free rate.
Risk-Neutral Valuation

- When the probability of up and down movements are \( p \) and \( 1-p \), the expected stock price at time \( T \) is \( S_0e^{rT} \).

- This is because the expected stock price at time \( T \) is:

\[
E(S_T) = pS_0u + (1 - p)S_0d
\]
\[
E(S_T) = \frac{e^{rT} - d}{u - d} S_0u + \left(1 - \frac{e^{rT} - d}{u - d}\right) S_0d
\]
\[
E(S_T) = \frac{S_0(ue^{rT} - ud + ud - de^{rT})}{u - d}
\]
\[
E(S_T) = S_0e^{rT}
\]
Risk-Neutral Valuation

- This shows that the stock price earns the risk-free rate.

- Binomial trees illustrate the general result that to value a derivative, we can assume that the expected return on the underlying asset is the risk-free rate, and that the discount rate is also the risk-free rate.

- This is known as using “risk-neutral valuation”.
Irrelevance of the Stock’s Expected Return

- When we are valuing an option in terms of the underlying stock, the expected return on the stock is irrelevant.

- The key reason is that the probabilities of future up or down movements are already incorporated into the price of the stock.
Original Example Revisited

Since $p$ is a risk-neutral probability:

$$20e^{0.12 \times 0.25} = 22p + 18(1 - p); \quad \text{so } p = 0.6523$$

Alternatively, we can use the formula:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$
Valuing the Call Option Using Risk-Neutral Valuation

The value of the Call option is:

\[
C = e^{-0.12 \times 0.25} \left[ 0.6523 \times 1 + 0.3477 \times 0 \right]
\]

\[C = 0.633\]
Valuing the Put Option Using Risk-Neutral Valuation

The value of the Put option is:

\[
P = e^{-0.12 \times 0.25} \left[ 0.6523 \times 0 + 0.3477 \times 3 \right]
\]

\[
P = 1.012
\]
Real World vs. Risk-Neutral World

- The probability of an up movement in a risk-neutral world is not the same as the probability of an up movement in the real world.

- Suppose that in the real world, the expected return on the stock is 16%, and $q$ is the probability of an up movement in the real world.

- We have: $22q + 18(1-q) = 20e^{0.16 \times 3/12}$ so that $q = 0.7041$

- The expected payoff from the option in the real world is then: $qx_1 + (1-q)x_0 = 0.7041$

- Unfortunately it is difficult to know the correct discount rate to apply to the expected payoff (It must be the option expected discount rate, higher than the stock’s)
A Two-Step Example

Figure 12.3, page 280

- Each time step is 3 months
- $K=21$, $r = 12\%$ (like before)
Valuing a Call Option

Figure 12.4, page 280

- Value at node B
  \[ = e^{-0.12 \times 0.25} (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257 \]
- Value at node A
  \[ = e^{-0.12 \times 0.25} (0.6523 \times 2.0257 + 0.3477 \times 0) \]
  \[ = 1.2823 \]
A Put Option Example

Figure 12.7, page 283

\[ K = 52, \text{ time step } = 1 \text{yr} \]

\[ r = 5\%, u = 1.2, d = 0.8, p = 0.6282 \]
What Happens When the Put Option is American  (Figure 12.8, page 284)

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<thead>
<tr>
<th>Node</th>
<th>Value</th>
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<tbody>
<tr>
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<td>5.0894</td>
</tr>
<tr>
<td>60</td>
<td>1.4147</td>
</tr>
<tr>
<td>40</td>
<td>12.0</td>
</tr>
<tr>
<td>C</td>
<td>12.0</td>
</tr>
<tr>
<td>48</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>48</td>
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<td>12.0</td>
</tr>
<tr>
<td>C</td>
<td>12.0</td>
</tr>
</tbody>
</table>

The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.
Delta

- Delta ($\Delta$) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.

- The value of $\Delta$ varies from node to node.

- It is the first of the “Greek Letters”, or measures of sensitivity of option prices with respect to the various underlying variables (stock price, interest rate, volatility, etc…).
Choosing $u$ and $d$

- One way of matching the volatility is to set:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where $\sigma$ is the volatility, and $\Delta t$ is the length of the time step.

- This is the approach used by Cox, Ross, and Rubinstein (1979)
Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures, the basic procedure for constructing the tree is the same except that the calculation of $p$ differs slightly.

- A continuous dividend yield $q$, a foreign risk-free rate $r_f$, or the domestic risk-free rate must be subtracted from the domestic risk-free rate in the formula for $p$. 
Generalizing the Probability $p$ of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a nondividend-paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where $q$ is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where $r_f$ is the foreign risk-free rate

$a = 1$ for a futures contract
Increasing the Time Steps

- In practice, at least 30 time steps are necessary to give good option values.

- But that minimum number of steps also depends on the option’s time to expiration.

- The software DerivaGem allows up to 500 time steps to be used.
The Black-Scholes-Merton Model

- The Black-Scholes-Merton model can be derived by looking at what happens to the price of a European call option as the time step tends to zero.

- For European options, the Black-Scholes price is the “binomial tree price” when the tree is taken to the limit with infinitely small steps.

- The outline of the proof can be found in the Appendix to Chapter 12.