The Greek Letters

Chapter 17
Example (Page 365)

- A bank has sold for $300,000 a European call option on 100,000 shares of a non-dividend-paying stock (the price of the option per share, “textbook style”, is therefore $3.00).

- $S_0 = 49, \ K = 50, \ r = 5\%, \ \sigma = 20\%, \ T = 20$ weeks, $\mu = 13\%$

- (Note that even though the price of the option does not depend on $\mu$, we list it here because it can impact the effectiveness of the hedge)

- The Black-Scholes-Merton value of the option is $240,000$ (or $2.40 for an option on one share of the underlying stock).

- How does the bank hedge its risk?
Naked & Covered Positions

- **Naked position:**
  - One strategy is to simply do nothing: taking no action and remaining exposed to the option risk.
  - In this example, since the firm wrote/sold the call option, it hopes that the stock remains below the strike price ($50) so that the option doesn’t get exercised by the other party and the firm keeps the premium (price of the call it received originally) in full.
  - But if the stock rises to, say, $60, the firm loses $(60-50) \times 100,000 = $1,000,000$ is much more than the $300,000$ received before.

- **Covered position:**
  - The firm can instead buy 100,000 shares today in anticipation of having to deliver them in the future if the stock rises.
  - If the stock declines, however, the option is not exercised but the stock position is hurt: if the stock goes down to $40, the loss is $(49-40) \times 100,000 = $900,000$. 

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Stop-Loss Strategy

- The stop-loss strategy is designed to ensure that the firm owns the stock if the option closes in-the-money and does not own the stock if the option closes out-of-the-money.

- The stop-loss strategy involves:
  - Buying 100,000 shares as soon as the stock price reaches $50.
  - Selling 100,000 shares as soon as the stock price falls below $50.
However, repeated transactions (costly) and the fact that you will always buy at $K+\varepsilon$ and sell at $K-\varepsilon$ makes this strategy not a very viable one.
Delta (See Figure 17.2, page 369)

- Delta ($\Delta$) is the rate of change of the option price with respect to the underlying.

![Diagram showing the relationship between option price and stock price with a slope of $\Delta = 0.6$.]
Hedge

- Trader would be hedged with the position:
  - Short/wrote 1000 options (each on one share)
  - buy 600 shares

- Gain/loss on the option position is offset by loss/gain on stock position

- Delta changes as stock price changes and time passes.

- Hedge position must therefore be rebalanced often, an example of *dynamic hedging.*

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Delta Hedging

- Delta hedging involves maintaining a delta neutral portfolio: a position with a delta of zero.

- The delta of a European call on a non-dividend-paying stock is $N(d_1)$

- The delta of a European put on the stock is $[N(d_1) - 1]$ or equivalently: $-N(-d_1)$. 
First Scenario for the Example:
Table 17.2 page 372

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost ('$000)</th>
<th>Cumulative Cost ($000)</th>
<th>Interest ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52,200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>48.12</td>
<td>0.458</td>
<td>(6,400)</td>
<td>(308.0)</td>
<td>2,252.3</td>
<td>2.2</td>
</tr>
<tr>
<td>2</td>
<td>47.37</td>
<td>0.400</td>
<td>(5,800)</td>
<td>(274.7)</td>
<td>1,979.8</td>
<td>1.9</td>
</tr>
<tr>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
<td>......</td>
</tr>
<tr>
<td>19</td>
<td>55.87</td>
<td>1.000</td>
<td>1,000</td>
<td>55.9</td>
<td>5,258.2</td>
<td>5.1</td>
</tr>
<tr>
<td>20</td>
<td>57.25</td>
<td>1.000</td>
<td>0</td>
<td>0</td>
<td>5263.3</td>
<td></td>
</tr>
</tbody>
</table>
First Scenario results

- At the end, the option is in-the-money.

- The buyer of the call option exercises it and pays the strike of 50 per share on the 100,000 shares held by the firm.

- This revenue of $5,000,000 partially offsets the cumulative cost of $5,263,300 and leaves a net cost of $263,300.
## Second Scenario for the Example

### Table 17.3 page 373

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost (‘$000)</th>
<th>Cumulative Cost ($000)</th>
<th>Interest ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52,200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>49.75</td>
<td>0.568</td>
<td>4,600</td>
<td>228.9</td>
<td>2,789.2</td>
<td>2.7</td>
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<td>2</td>
<td>52.00</td>
<td>0.705</td>
<td>13,700</td>
<td>712.4</td>
<td>3,504.3</td>
<td>3.4</td>
</tr>
<tr>
<td></td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>19</td>
<td>46.63</td>
<td>0.007</td>
<td>(17,600)</td>
<td>(820.7)</td>
<td>290.0</td>
<td>0.3</td>
</tr>
<tr>
<td>20</td>
<td>48.12</td>
<td>0.000</td>
<td>(700)</td>
<td>(33.7)</td>
<td>256.6</td>
<td></td>
</tr>
</tbody>
</table>
Second Scenario results

- At the end, the option is out-of-the-money.
- The buyer of the call option does not exercise it and nothing happens on that front.
- The delta having gone to zero, the firm has no shares left.
- The net cumulative cost of hedging is $256,600.
Comments on dynamic hedging

- Since by hedging the short/written call the firm was essentially *synthetically replicating* a long position in the option, the (PV of the) cost of the replication should be close to the Black-Scholes price of the call: $240,000.

- They differ a little in reality because the hedge was rebalanced once a week only, instead of continuously.

- Also, in reality, volatility may not be constant, and there are some transaction costs.
The concept of delta is not limited to one security.

The delta of a portfolio of options or derivatives dependent on a single asset with price $S$ is given by $\Delta_{\Pi} = \Delta \Pi / \Delta S$.

If a portfolio of $n$ options consists of a quantity $q_i$ of option $i$, the delta of the portfolio is given by:

$$\Delta_{\Pi} = \sum_{i=1}^{n} q_i \Delta_i$$
Delta of a Portfolio: example

- Suppose a bank has the following 3 option positions:
  - A long position in 100,000 call options with a delta of 0.533 for each.
  - A short position in 200,000 call options with a delta of 0.468 for each.
  - A short position in 50,000 put options with a delta of -0.508 for each.

- The delta of the whole portfolio is:
  \[ \Delta_\Pi = 100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) \]
  \[ \Delta_\Pi = -14,900 \]

- The portfolio can thus be made delta neutral by buying 14,900 shares of the underlying stock.
- It also means that when the stock goes up by $1, the portfolio goes down by $14,900.
Theta

- The Theta ($\Theta$) of a derivative (or portfolio of derivatives) is the rate of change of its value with respect to the passage of time.

- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines.

- It is sometimes referred to as the time decay of the option.
Theta

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

\[
\Theta(\text{call}) = -S_0 N'(d_1) \sigma \frac{1}{2\sqrt{T}} - rK e^{-rT} N(d_2)
\]

with \( N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \)

\[
\Theta(\text{put}) = -S_0 N'(d_1) \sigma \frac{1}{2\sqrt{T}} + rK e^{-rT} N(-d_2)
\]
Theta for Call Option:

\[ K=50, \quad \sigma = 25\%, \quad r = 5\%, \quad T = 1 \]

Note that theta is here expressed per calendar day. The theta obtained from the formula is annual. For example, for \( S=50 \), \( \theta = -3.6252 \) and so we have \( \theta = -3.6252/365 = -0.01 \) per calendar day.
Gamma

- Gamma ($\Gamma$) is the rate of change of delta ($\Delta$) with respect to the price of the underlying asset, so it is the “change of the change”.

- Gamma is greatest for options that are close to being at-the-money since this is where the slope, delta, changes the most.

- If Gamma is high, then delta changes rapidly.

- If Gamma is low, then delta changes slowly.

- Gamma is the second derivative of the option price with respect to the stock: it can be stated as $\Gamma = d^2C/dS^2$ or $\Gamma = d^2P/dS^2$. 
Gamma

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

\[ \Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \]

with \( N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \)

- The formula is valid for both call and put options.
Gamma for Call or Put Option: 
\( K=50, \sigma = 25\%, r = 5\%, T = 1 \)
Gamma Addresses Delta Hedging
Errors Caused By Curvature
(Figure 17.7, page 377)
Interpretation of Gamma

- For a delta neutral portfolio,
  \[ \Delta \Pi \approx \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2 \]

Positive Gamma

Negative Gamma
Relationship Between Delta, Gamma, and Theta

For a portfolio \( \Pi \) of derivatives on a non-dividend-paying stock, we have:

\[
\Theta + rS_0 \Delta + \frac{1}{2} \sigma^2 S_0^2 \Gamma = r \Pi
\]

or \[
\frac{\partial C}{\partial t} + rS_0 \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 C}{\partial S^2} = rC \text{ if the portfolio consists of one Call only}
\]

Solving this partial differential equation for \( C \) (or \( P \) if a Put) yields the Black-Scholes pricing equation, by the way.
How to make a portfolio “Gamma Neutral”

- Recall that, just like for any other Greek (Delta,…), the Gamma of a portfolio $\Pi$ is given by: $\Gamma_\Pi = n_1 \Gamma_1 + n_2 \Gamma_2 + n_3 \Gamma_3 \ldots$
- One can thus make the total Gamma equal to zero with the appropriate addition of a certain number of options with a certain Gamma.

- As an example, assume your position is currently Delta neutral ($\Delta=0$) but that $\Gamma=-3,000$. You however find a call option with $\Delta_C=0.62$ and $\Gamma_C=1.50$
- You need $\Gamma_\Pi = (1)\Gamma_{\text{existing position}} + n_C \Gamma_C = 0$ i.e. need $-3,000 + n_C (1.50) = 0$.
- Buy $n_C = 3,000/1.5 = 2,000$ Call options and now have a Gamma of zero.

- The only issue is that the addition of the call options shifted your delta.
- Your new portfolio delta is $\Delta_\Pi = (1)0 + 2,000\Delta_C = 2,000(0.62) = 1,240$.
- Therefore 1,240 shares of the underlying asset must be sold from the portfolio in order to keep it delta neutral. It is now Delta-Gamma neutral.
Vega (not an actual Greek letter, but it sounds like one: good enough)

- Vega ($\nu$) is the rate of change of the value of a derivative (or a derivatives portfolio) with respect to the volatility of the underlying asset.

- Vega is an important measure because in practice, the volatility $\sigma$ is not constant and changes over time.

- If Vega is highly positive or negative, the portfolio’s value is very sensitive to small changes in volatility.

- If Vega is close to zero, volatility changes have almost no impact on the value of the portfolio.
Vega

- For European options on a non-dividend-paying stock, it can be shown from the Black-Scholes formulas that:

\[ \nu = S_0 \sqrt{T} N'(d_1) \]

with \( N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \)

- The formula is valid for both call and put options.
Vega for a Call or Put Option: 
\[ K=50, \sigma = 25\%, r = 5\%, T = 1 \]
Managing Delta, Gamma, & Vega risk \textit{all at once}

- We know that Delta can be changed by taking a position in the underlying asset.

- We also know that to adjust Gamma and Vega, it is necessary to take a position in an option or other derivative.

- However, if only one other derivative is added, either the Gamma risk or the Vega risk will be canceled, but not both at the same time (except by some coincidence).

- So we need to add \textit{two new derivatives} to hedge all risks.
What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral?

**Answer:** Long 10,000 options, short 6,000 of the asset.

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral?

**Answer:** Long 4,000 options, short 2,400 of the asset.
What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral all at once?

We solve
\[
-5000 + 0.5w_1 + 0.8w_2 = 0 \\
-8000 + 2.0w_1 + 1.2w_2 = 0
\]

to get \( w_1 = 400 \) and \( w_2 = 6,000 \).
We therefore require long positions of 400 and 6,000 in option 1 and option 2.

However, because these additions result in an incremental positive delta of 400(0.6)+6,000(0.5)=3,240, we also need to take a short position of 3,240 in the asset in order to also make the portfolio delta neutral.
Rho

Rho is the rate of change of the value of a derivative with respect to the interest rate.

It is usually small and not a big issue in practice, unless the option is deep in-the-money and has a long horizon (discounting a larger cash flow over a longer horizon is more relevant then).
Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day.

- Whenever the opportunity arises, they improve gamma and vega.

- As the portfolio becomes larger, hedging becomes less expensive since the trading cost per option goes down.
Scenario Analysis

- In addition to monitoring risks such as delta, gamma, and vega, option traders often also conduct a scenario analysis.

- A scenario analysis involves computing the gains and losses on the portfolio over a specified period of time under a variety of different scenarios.

- Often the two main sources of risk looked as variables (the scenarios) are the underlying asset price and volatility.
Greek Letters for European Options on an Asset that Provides a (dividend) Yield at Rate $q$ (Table 17.6, page 386)

<table>
<thead>
<tr>
<th>Greek Letter</th>
<th>Call Option</th>
<th>Put Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta</td>
<td>$e^{-qT} N(d_1)$</td>
<td>$e^{-qT}[N(d_1) - 1]$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$</td>
<td>$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$</td>
</tr>
<tr>
<td>Theta</td>
<td>$-S_0N'(d_1)\sigma e^{-qT} / (2\sqrt{T})$</td>
<td>$-S_0N'(d_1)\sigma e^{-qT} / (2\sqrt{T})$</td>
</tr>
<tr>
<td></td>
<td>$+qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$</td>
<td>$-qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$</td>
</tr>
<tr>
<td>Vega</td>
<td>$S_0\sqrt{T} N'(d_1)e^{-qT}$</td>
<td>$S_0\sqrt{T} N'(d_1)e^{-qT}$</td>
</tr>
<tr>
<td>Rho</td>
<td>$KTe^{-rT} N(d_2)$</td>
<td>$-KTe^{-rT} N(-d_2)$</td>
</tr>
</tbody>
</table>
Using Futures for Delta Hedging

- The delta of a futures contract on an asset paying a yield at rate $q$ is $e^{(r-q)T}$, since we know that $F = Se^{(r-q)T}$.

- The position required in futures for delta hedging (instead of using the spot asset) is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot asset.
Example of Using Futures for Delta Hedging (instead of the spot)

- A portfolio of currency options held by a US bank can be made delta neutral with a short position of 458,000 pounds sterling (of the spot asset, if it were used).

- If the US riskless rate is 4% and the UK rate 7%, hedging for 9 months using a short position in futures contracts instead of shorting the spot currency would require shorting:
  \[ e^{-(0.04-0.07) \times 9/12} \times 458,000 \text{ or £468,442 in futures contracts.} \]

- Since each futures contract is for £62,500 the number of contracts to be shorted is 468,442/62,500 = 7 contracts.
Hedging vs. Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc…

- When we create an option synthetically we take positions that match delta, gamma, vega, etc…
In October of 1987, many portfolio managers attempted to create a put option on a portfolio synthetically. This involves initially selling enough of the portfolio (or of index futures) to match the $\Delta$ of the put option.
Portfolio Insurance (continued)

- As the value of the portfolio increases, the $\Delta$ of the put becomes less negative and some of the original portfolio is repurchased.

- As the value of the portfolio decreases, the $\Delta$ of the put becomes more negative and more of the portfolio must be sold.

- The strategy did not work well on October 19, 1987 because since everyone was doing the same thing, liquidity became an issue.

- Additionally, investors that anticipated the portfolio insurers reaction sold their positions as well, exacerbating the problem and precipitating the price decrease, making the crash worse.