RISK AND RISK AVERSION

The investment process consists of two broad tasks. One task is security and market analysis, by which we assess the risk and expected-return attributes of the entire set of possible investment vehicles. The second task is the formation of an optimal portfolio of assets. This task involves the determination of the best risk-return opportunities available from feasible investment portfolios and the choice of the best portfolio from the feasible set. We start our formal analysis of investments with this latter task, called portfolio theory. We return to the security analysis task in later chapters. This chapter introduces three themes in portfolio theory, all centering on risk. The first is the basic tenet that investors avoid risk and demand a reward for engaging in risky investments. The reward is taken as a risk premium, the difference between the expected rate of return and that available on alternative risk-free investments. The second theme allows us to quantify investors' personal trade-offs between portfolio risk and expected return. To do this we introduce the utility function, which assumes that investors can assign a welfare or "utility" score to any investment portfolio depending on its risk and return. Finally, the third fundamental principle is that we cannot evaluate the risk of an asset separate from the portfolio of which it is a part; that is, the proper way to measure the risk of an individual asset is to assess its impact on the volatility of the entire portfolio of investments. Taking this approach, we find that seemingly risky securities may be portfolio stabilizers and actually low-risk assets. Appendix A to this chapter describes the theory and practice of measuring portfolio risk by the variance or standard deviation of returns. We discuss other potentially relevant characteristics of the probability distribution.
of portfolio returns, as well as the circumstances in which variance is sufficient to measure risk. Appendix B discusses the classical theory of risk aversion.

6.1 RISK AND RISK AVERSION

Risk with Simple Prospects

The presence of risk means that more than one outcome is possible. A simple prospect is an investment opportunity in which a certain initial wealth is placed at risk, and there are only two possible outcomes. For the sake of simplicity, it is useful to elucidate some basic concepts using simple prospects.1

Take as an example initial wealth, $W_0$, of $100,000$, and assume two possible results. With a probability $p = 0.6$, the favorable outcome will occur, leading to final wealth $W_1 = 150,000$. Otherwise, with probability $1 - p = 0.4$, a less favorable outcome. $W_2 = 80,000$, will occur. We can represent the simple prospect using an event tree:

\[
\begin{align*}
W &= 100,000 \\
W_1 &= 150,000 \\
W_2 &= 80,000
\end{align*}
\]

Suppose an investor is offered an investment portfolio with a payoff in one year described by a simple prospect. How can you evaluate this portfolio?

First, try to summarize it using descriptive statistics. For instance, the mean or expected end-of-year wealth, denoted $E(W)$, is:

\[
E(W) = pW_1 + (1 - p)W_2 = (0.6 \times 150,000) + (0.4 \times 80,000) = 122,000
\]

The expected profit on the $100,000 investment portfolio is $22,000: 122,000 - 100,000.

The variance, $\sigma^2$, of the portfolio’s payoff is calculated as the expected value of the squared deviation of each possible outcome from the mean:

\[
\sigma^2 = p[W_1 - E(W)]^2 + (1 - p)[W_2 - E(W)]^2 = 0.6(150,000 - 122,000)^2 + 0.4(80,000 - 122,000)^2 = 1,176,000,000
\]

The standard deviation, $\sigma$, which is the square root of the variance, is therefore $\sqrt{1,176,000,000} = 34,292.86$.

Clearly, this is risky business. The standard deviation of the payoff is large, much larger than the expected profit of $22,000. Whether the expected profit is large enough to justify such risk depends on the alternative portfolios.

1 Chapters 6 through 8 rely on some basic results from elementary statistics. For a refresher, see the Quantitative Review in the Appendices at the end of the book.
Let us suppose Treasury bills are one alternative to the risky portfolio. Suppose that at the time of the decision, a one-year T-bill offers a rate of return of 5%. $100,000 can be invested to yield a sure profit of $5,000. We can now draw the decision tree.

\[
\begin{array}{c}
\$100,000 \quad p = 6 \\
A. \text{Invest in risky prospect} \quad \frac{\text{profit} = \$50,000}{1 - p = 4} \quad \text{profit} = -\$20,000 \\
B. \text{Invest in risk-free T-bill} \quad \text{profit} = \$5,000
\end{array}
\]

Earlier we showed the expected profit on the prospect to be $22,000. Therefore, the expected marginal, or incremental, profit of the risky portfolio over investing in safe T-bills is $22,000 - $5,000 = $17,000, meaning that one can earn a risk premium of $17,000 as compensation for the risk of the investment.

The question of whether a given risk premium provides adequate compensation for an investment’s risk is age-old. Indeed, one of the central concerns of finance theory (and much of this text) is the measurement of risk and the determination of the risk premiums that investors can expect of risky assets in well-functioning capital markets.

**Concept Check**

*Risk, Speculation, and Gambling*

One definition of speculation is "the assumption of considerable business risk in obtaining commensurate gain." Although this definition is fine linguistically, it is useless without first specifying what is meant by "commensurate gain" and "considerable risk."

By "commensurate gain" we mean a positive risk premium, that is, an expected profit greater than the risk-free alternative. In our example, the dollar risk premium is $17,000, the incremental expected gain from taking on the risk. By "considerable risk" we mean that the risk is sufficient to affect the decision. An individual might reject a prospect that has a positive risk premium because the added gain is insufficient to make up for the risk involved.

To gamble is "to bet or wager on an uncertain outcome." If you compare this definition to that of speculation, you will see that the central difference is the lack of "commensurate gain." Economically speaking, a gamble is the assumption of risk for no purpose but enjoyment of the risk itself, whereas speculation is undertaken in spite of the risk involved because one perceives a favorable risk-return trade-off. To turn a gamble into a speculative prospect requires an adequate risk premium to compensate risk-averse investors for the risks they bear. Hence, risk aversion and speculation are not inconsistent.

In some cases a gamble may appear to the participant as speculation. Suppose two investors disagree sharply about the future exchange rate of the U.S. dollar against the British pound. They only choose to bet on the outcome. Suppose that Paul will pay Mary $100 if the value of £1 exceeds $1.70 one year from now, whereas Mary will pay Paul if the pound is worth less than $1.70. There are only two relevant outcomes: (1) the pound will exceed $1.70, or (2) it will fall below $1.70. If both Paul and Mary agree on the probabilities of the two possible outcomes, and if neither party anticipates a loss, it must be that they assign
CHAPTER 6: Risk and Risk Aversion

\[ p = 0.5 \] to each outcome. In that case the expected profit is both zero and each has entered one side of a gambling prospect.

What is more likely, however, is that the bet results from differences in the probabilities that Paul and Mary assign to the outcome. Mary assigns it \( p > 0.5 \), whereas Paul’s assessment is \( p < 0.5 \). They perceive, subjectively, two different prospects. Economists call this case of differing beliefs “heterogeneous expectations.” In such cases investors on each side of a financial position see themselves as speculating rather than gambling.

Both Paul and Mary should be asking, "Why is the other willing to invest in the side of a risky prospect that I believe offers a negative expected profit?" The ideal way to resolve heterogeneous beliefs is for Paul and Mary to "merge their information," that is, for each party to verify that he or she possesses all relevant information and processes the information properly. Of course, the acquisition of information and the extensive communication that is required to eliminate all heterogeneity in expectations is costly, and thus up to a point heterogeneous expectations cannot be taken as irrational. If, however, Paul and Mary enter such contracts frequently, they would recognize the information problem in one of two ways: Either they will realize that they are creating gambles when each wins half of the bets, or the consistent loser will admit that he or she has been betting on the basis of inferior forecasts.

Assume that dollar-denominated T-bills in the United States and pound-denominated bills in the United Kingdom offer equal yields at maturity. Both are short-term assets, and both are free of default risk. Neither offers investors a risk premium. However, a U.S. investor who holds U.K. bills is subject to exchange rate risk, because the pounds earned on the U.K. bills eventually will be exchanged for dollars at the future exchange rate. Is the U.S. investor engaging in speculation or gambling?

**Risk Aversion and Utility Values**

We have discussed risk with simple prospects and how risk premiums bear on speculation. A prospect that has a zero risk premium is called a fair game. Investors who are risk aversive reject investment portfolios that are fair games or worse. Risk-averse investors are willing to consider only risk-free or speculative prospects with positive risk premia. Loosely speaking, a risk-averse investor "penalizes" the expected rate of return of a risky portfolio by a certain percentage (or penalizes the expected profit by a dollar amount) to account for the risk involved. The greater the risk, the larger the penalty. One might wonder why we assume risk aversion as fundamental. We believe that most investors would accept this view from simple introspection, but we discuss the question more fully in Appendix B of this chapter.

We can formalize the notion of a risk-preference system. To do so, we will assume that each investor can assign a welfare, or utility, score to competing investment portfolios based on the expected return and risk of those portfolios. The utility score may be viewed as a means of ranking portfolios. Higher utility values are assigned to portfolios with more attractive risk-return profiles. Portfolios receive higher utility scores for higher expected returns and lower scores for higher volatility. Many particular "scoring" systems are legitimate. One reasonable function that is commonly employed by financial theorists and the AIMR (Association of Investment Management and Research) assigns a portfolio with expected return \( E(r) \) and variance of returns \( \sigma^2 \) the following utility score:

\[ U = E(r) - 0.05\sigma^2 \]

where \( U \) is the utility value and \( A \) is an index of the investor’s risk aversion. The factor of 0.05 is a scaling convention that allows us to express the expected return and standard deviation in equation 6.1 as percentages rather than decimals.
What four-letter word should pop into mind when the stock market takes a harrowing nose dive? No, not those. R.A.S.E.K.

Risk is the potential for realizing low returns or even losing money, possibly preventing you from meeting important objectives, like sending your kids to the college of your choice or having the retirement lifestyle you crave.

But many financial advisers and other experts say that these days investors aren’t taking the idea of risk as seriously as they should, and they are overexpending themselves to stocks.

“The market has been as good for years that investors no longer believe there is risk in investing,” says Gary Schatsky, a financial adviser in New York.

So before the market goes down and stays down, be sure that you understand your tolerance for risk and that your portfolio is designed to match it.

Assessing your risk tolerance, however, can be tricky. You must consider not only how much risk you can afford to take but also how much risk you can stand to take.

Determining how much risk you can stand—your temperamental tolerance for risk—is more difficult. It isn’t quantitative.

To that end, many financial advisers, brokerage firms and mutual-fund companies have created risk quizzes to help people determine whether they are conservative, moderate or aggressive investors. Some firms that offer such quizzes include Merrill Lynch, T. Rowe Price Associates Inc., Baltimore; Zurich Group Inc.’s Scudder Kemper Investments Inc., New York, and Vanguard Group in Malvern, Pa.

Typically, risk questionnaires include seven to 10 questions about a person’s investing experience, financial security and tendency to make risky or conservative choices.

The benefit of the questionnaires is that they are an objective resource people can use to get at least a rough idea of their risk tolerance. “It’s impossible for someone to assess their risk tolerance alone,” says Mr. Bernstein. “I may say I don’t like risk, yet will take more risk than the average person.”

Many experts warn, however, that the questionnaires should be used simply as a first step to assessing risk tolerance. “They are not precise,” says Ron Meier, a certified public accountant.

The second step, many experts agree, is to ask yourself some difficult questions, such as: How much you can stand to lose over the long term? “Most people can stand to lose a heck of a lot temporarily,” says Mr. Schatsky. The real acid test, he says, is: How much of your portfolio value you can stand to lose over months or years.

As it turns out, most people rank as middle-of-the-road risk-takers, say several advisers. “Only about 10% to 15% of my clients are aggressive,” says Mr. Roget.

What’s Your Risk Tolerance?

Circle the letter that corresponds to your answer.

1. Just 60 days after you put money into an investment, its price falls 20%. Assuming none of the fundamentals have changed, what would you do?
   a. Sell to avoid further worry and try something else.
   b. Do nothing and wait for the investment to come back.
   c. Buy more. It was a good investment before; now it’s a cheap investment too.

2. Now look at the previous question another way. Your investment fell 20%, but it’s part of a portfolio being used to meet investment goals with three different time horizons.

Equation 6.1 is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk. Whether variance is an adequate measure of portfolio risk is discussed in Appendix A. The extent to which variance lowers utility depends on the investor’s degree of risk aversion. More risk-averse investors (who have the larger AV) penalize risky investments more severely. Investors choosing among competing investment portfolios will select the one providing the highest utility level.

Risk aversion obviously will have a major impact on the investor’s appropriate risk-return trade-off. The above box discusses some techniques that financial advisers use to gauge the risk aversion of their clients.

Notice in equation 6.1 that the utility provided by a risk-free portfolio is simply the rate of return on the portfolio, because there is no penalization for risk. This provides us with a
2A. What would you do if the goal were five years away?
   a. Sell
   b. Do nothing
   c. Buy more

2B. What would you do if the goal were 15 years away?
   a. Sell
   b. Do nothing
   c. Buy more

2C. What would you do if the goal were 30 years away?
   a. Sell
   b. Do nothing
   c. Buy more

3. The price of your retirement investment jumps 25% a month after you buy it. Again, the fundamentals haven’t changed. After you finish gloating, what do you do?
   a. Sell it and lock in your gains
   b. Stay put and hope for more gains
   c. Buy more; it could go higher

4. You’re investing for retirement, which is 15 years away. Which would you rather do?
   a. Invest in a money-market fund or guaranteed investment contract, giving up the possibility of major gains, but virtually assuring the safety of your principal
   b. Invest in a 50-50 mix of bond funds and stock funds, in hopes of getting some growth, but also giving yourself some protection in the form of steady income
   c. Invest in aggressive growth mutual funds whose value will probably fluctuate significantly during the year, but have the potential for impressive gains over five or 10 years

5. You just won a big prize! But which one? It’s up to you:
   a. $2,000 in cash
   b. A 50% chance to win $5,000
   c. A 20% chance to win $15,000

6. A good investment opportunity just came along. But you have to borrow money to get it. Would you take out a loan?
   a. Definitely not
   b. Perhaps
   c. Yes

7. Your company is selling stock to its employees. In three years, management plans to take the company public. Until then, you won’t be able to sell your shares and you will get no dividends. But your investment could multiply as much as 10 times when the company goes public. How much money would you invest?
   a. None
   b. Two months’ salary
   c. Four months’ salary

Scoring Your Risk Tolerance

To score the quiz, add up the number of answers you gave in each category and then multiply as shown to find your score:

(a) answers ——— x 1 = _______ points
(b) answers ——— x 2 = _______ points
(c) answers ——— x 3 = _______ points

YOUR SCORE _______ points

If you scored . . . You may be a:
9-14 points Conservative investor
15-21 points Moderate investor
22-27 points Aggressive investor


convenient benchmark for evaluating portfolios. For example, recall the earlier investment problem, choosing between a portfolio with an expected return of 22% and a standard deviation of 34% and T-bills providing a risk-free rate of 5%. Although the risk premium on the risky portfolio is large, 17%, the risk of the project is so great that an investor would not need to be very risk averse to choose the safe all-bills strategy. Even for $A = 3$, a moderate risk-aversion parameter, equation 6.1 shows the risky portfolio’s utility value as $22 - 1.005 \times 3 \times 34^2 = 4.66\%$, which is slightly lower than the risk-free rate. In this case, one would reject the portfolio in favor of T-bills.

The downward adjustment of the expected return as a penalty for risk is $0.005 \times 3 \times 34^2 = 17.34\%$. If the investor were less risk averse (more risk tolerance), for example, with $A = 2$, she would adjust the expected rate of return downward by only 11.56%. In that case the
utility level of the portfolio would be 10.44%, higher than the risk-free rate, leading her to accept the prospect.

A portfolio has an expected rate of return of 20% and standard deviation of 20%. Bills offer a sure rate of return of 7%. Which investment alternative will be chosen by an investor whose \( A = 4? \)

What if \( A = 8? \)

Because we can compare utility values to the rate offered on risk-free investments when choosing between a risky portfolio and a safe one, we may interpret a portfolio’s utility value as its “certainty equivalent” rate of return to an investor. That is, the certainty equivalent rate of a portfolio is the rate that risk-free investments would need to offer with certainty to be considered equally attractive as the risky portfolio.

Now we can say that a portfolio is desirable only if its certainty equivalent rate exceeds that of the risk-free alternative. A sufficiently risk-averse investor may assign any risky portfolio, even one with a positive risk premium, a certainty equivalent rate of return that is below the risk-free rate, which will cause the investor to reject the portfolio. At the same time, a less risk-averse (more risk-tolerant) investor may assign the same portfolio a certainty equivalent rate that exceeds the risk-free rate and thus will prefer the portfolio to the risk-free alternative. If the risk premium is zero or negative to begin with, any downward adjustment to utility only makes the portfolio look worse. Its certainty equivalent rate will be below that of the risk-free alternative for all risk-averse investors.

In contrast to risk-averse investors, risk-neutral investors judge risky prospects solely by their expected rates of return. The level of risk is irrelevant to the risk-neutral investor, meaning that there is no penalization for risk. For this investor a portfolio’s certainty equivalent rate is simply its expected rate of return.

A risk lover is willing to engage in fair games and gambles; this investor adjusts the expected return upward to take into account the “fun” of confronting the prospect’s risk. Risk lovers will always take a fair game because their upward adjustment of utility for risk gives the fair game a certainty equivalent that exceeds the alternative of the risk-free investment.

We can depict the individual’s trade-off between risk and return by plotting the characteristics of potential investment portfolios that the individual would view as equally...
attractive on a graph with axes measuring the expected value and standard deviation of portfolio returns. Figure 6.1 plots the characteristics of one portfolio.

Portfolio $P$, which has expected return $E(r_p)$ and standard deviation $\sigma_p$, is preferred by risk-averse investors to any portfolio in quadrant IV because it has an expected return equal to or greater than any portfolio in that quadrant and a standard deviation equal to or smaller than any portfolio in that quadrant. Conversely, any portfolio in quadrant I is preferable to portfolio $P$ because its expected return is equal to or greater than $P$'s and its standard deviation is equal to or smaller than $P$'s.

This is the mean-standard deviation, or equivalently, mean-variance (M-V) criterion. It can be stated as: $A$ dominates $B$ if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

and at least one inequality is strict (rules-out the equality).

In the expected return–standard deviation plane in Figure 6.1, the preferred direction is northwest, because in this direction we simultaneously increase the expected return and decrease the variance of the rate of return. This means that any portfolio that lies northwest of $P$ is superior to $P$.

What can be said about portfolios in the quadrants II and III? Their desirability, compared with $P$, depends on the exact nature of the investor’s risk aversion. Suppose an investor identifies all portfolios that are equally attractive as portfolio $P$. Starting at $P$, an increase in standard deviation lowers utility; it must be compensated for by an increase in expected return. Thus point $Q$ in Figure 6.2 is equally desirable to this investor as $P$. Investors will be equally attracted to portfolios with high risk and high expected returns compared with other portfolios with lower risk but lower expected returns. These equally preferred portfolios will lie in the mean–standard deviation plane on a curve that connects all portfolio points with the same utility value (Figure 6.2), called the indifference curve.

To determine some of the points that appear on the indifference curve, examine the utility values of several possible portfolios for an investor with $A = 4$, presented in Table 6.1.

---

**Figure 6.2**
The indifference curve.
Table 6.1  Utility Values of Possible Portfolios for Investor with Risk Aversion, A = 4

<table>
<thead>
<tr>
<th>Expected Return, E(r)</th>
<th>Standard Deviation, σ</th>
<th>Utility = E(r) - .005Aσ²</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>20.0%</td>
<td>10 - .005 * 4 * 400 = 2</td>
</tr>
<tr>
<td>15</td>
<td>25.5</td>
<td>15 - .005 * 4 * 650 = 2</td>
</tr>
<tr>
<td>20</td>
<td>30.0</td>
<td>20 - .005 * 4 * 900 = 2</td>
</tr>
<tr>
<td>25</td>
<td>34.9</td>
<td>25 - .005 * 4 * 1,150 = 2</td>
</tr>
</tbody>
</table>

Note that each portfolio offers identical utility, because the high-return portfolios also have high risk.

CONCEPT CHECK QUESTION 4

a. How will the indifference curve of a less risk-averse investor compare to the indifference curve drawn in Figure 6.2?
b. Draw both indifference curves passing through point P.

6.2 PORTFOLIO RISK

Asset Risk versus Portfolio Risk

Investor portfolios are composed of diverse types of assets. In addition to direct investment in financial markets, investors have stakes in pension funds, life insurance policies with savings components, homes, and not least, the earning power of their skills (human capital).

Investors must take account of the interplay between asset returns when evaluating the risk of a portfolio. At a most basic level, for example, an insurance contract serves to reduce risk by providing a large payoff when another part of the portfolio is faring poorly. A fire insurance policy pays off when another asset in the portfolio—a house or factory, for example—suffers a big loss in value. The offsetting pattern of returns on these two assets (the house and the insurance policy) stabilizes the risk of the overall portfolio. Investing in an asset with a payoff pattern that offsets exposure to a particular source of risk is called hedging.

Insurance contracts are obvious hedging vehicles. In many contexts financial markets offer similar, although perhaps less direct, hedging opportunities. For example, consider two firms, one producing suntan lotion, the other producing umbrellas. The shareholders of each firm face weather risk of an opposite nature. A rainy summer lowers the return on the suntan-lotion firm but raises it on the umbrella firm. Shares of the umbrella firm act as "weather insurance" for the suntan-lotion firm shareholders in the same way that fire insurance policies insure houses. When the lotion firm does poorly (bad weather), the "insurance" asset (umbrella shares) provides a high payoff that offsets the loss.

Another means to control portfolio risk is diversification, whereby investments are made in a wide variety of assets so that exposure to the risk of any particular security is limited. By placing one's eggs in many baskets, overall portfolio risk actually may be less than the risk of any component security considered in isolation.

To examine these effects more precisely, and to lay a foundation for the mathematical properties that will be used in coming chapters, we will consider an example with less than perfect hedging opportunities, and in the process review the statistics underlying portfolio risk and return characteristics.
A Review of Portfolio Mathematics

Consider the problem of Humanex, a nonprofit organization deriving most of its income from the return on its endowment. Years ago, the founders of Best Candy wised a large block of Best Candy stock to Humanex with the provision that Humanex may never sell it. This block of shares now comprises 50% of Humanex's endowment. Humanex has free choice as to where to invest the remainder of its portfolio.\(^1\)

The value of Best Candy stock is sensitive to the price of sugar. In years when world sugar crops are low, the price of sugar rises significantly and Best Candy suffers considerable losses.

We can describe the fortunes of Best Candy stock using the following scenario analysis:

<table>
<thead>
<tr>
<th>Normal Year for Sugar</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Bullish</strong> Stock Market</td>
<td><strong>Bearish</strong> Stock Market</td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
</tr>
<tr>
<td>Rate of return</td>
<td>25%</td>
</tr>
</tbody>
</table>

To summarize these three possible outcomes using conventional statistics, we review some of the key rules governing the properties of risky assets and portfolios.

**Rule 1** The mean or expected return of an asset is a probability-weighted average of its return in all scenarios. Calling \(Pr(i)\) the probability of scenario \(i\) and \(r(i)\) the return in scenario \(i\), we may write the expected return, \(E(r)\), as

\[
E(r) = \sum Pr(i) r(i) \tag{6.2}
\]

Applying this formula to the case at hand, with three possible scenarios, we find that the expected rate of return of Best Candy's stock is

\[
E(r_{\text{mean}}) = (0.5 \times 25) + (0.3 \times 10) + 2(-25) = 10.5\% \tag{6.2}
\]

**Rule 2** The variance of an asset's returns is the expected value of the squared deviations from the expected return. Symbolically,

\[
\sigma^2 = \sum Pr(i)[r(i) - E(r)]^2 \tag{6.3}
\]

Therefore, in our example

\[
\sigma_{\text{mean}}^2 = 0.5(25 - 10.5)^2 + 0.3(10 - 10.5)^2 + 0.2(-25 - 10.5)^2 = 357.25
\]

The standard deviation of Best's return, which is the square root of the variance, is

\[
\sqrt{357.25} = 18.9\%
\]

Humanex has 50% of its endowment in Best's stock. To reduce the risk of the overall portfolio, \(k\) it could invest the remainder in T-bills, which yield a sure rate of return of 5%. To derive the return of the overall portfolio, we apply rule 3.

**Rule 3** The rate of return on a portfolio is a weighted average of the rates of return of each asset comprising the portfolio, with portfolio proportions as weights. This implies that the expected rate of return on a portfolio is a weighted average of the expected rate of return on each component asset.

\(^1\) The portfolio is admittedly informal. We use this example only to illustrate the various concepts that might be useful to control risk and to review some useful results from statistics.
Humanex's portfolio proportions in each asset are .5, and the portfolio's expected rate of return is

\[ E(r_{\text{Humanex}}) = 0.5E(r_{\text{Best}}) + 0.5E(r_{\text{Bills}}) = (0.5 \times 10.5) + (0.5 \times 5) = 7.75\% \]

The standard deviation of the portfolio may be derived from rule 4.

Rule 4 When a risky asset is combined with a risk-free asset, the portfolio standard deviation equals the risky asset's standard deviation multiplied by the portfolio proportion invested in the risky asset.

The Humanex portfolio is 50% invested in Best stock and 50% invested in risk-free bills. Therefore,

\[ \sigma_{\text{Humanex}} = 0.5\sigma_{\text{Best}} = 0.5 \times 18.9 = 9.45\% \]

By reducing its exposure to the risk of Best by half, Humanex reduces its portfolio standard deviation by half. The cost of this risk reduction, however, is a reduction in expected return. The expected rate of return on Best stock is 10.5%. The expected return on the one-half T-bill portfolio is 7.75%. Thus, while the risk premium for Best stock over the 5% rate on risk-free bills is 5.5%, it is only 2.75% for the half T-bill portfolio. By reducing the share of Best stock in the portfolio by one-half, Humanex reduces its portfolio risk premium by one-half, from 5.5% to 2.75%.

In an effort to improve the contribution of the endowment to the operating budget, Humanex's trustees hire Sally, a recent MBA, as a consultant. Researching the sugar and candy industry, Sally discovers, not surprisingly, that during years of sugar shortage, SugarKane, a big Hawaiian sugar company, reaps unusual profits and its stock price soars. A scenario analysis of SugarKane's stock looks like this:

<table>
<thead>
<tr>
<th>Normal Year for Sugar</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullish Stock Market</td>
<td>Bearish Stock Market</td>
</tr>
<tr>
<td>Probability</td>
<td>.5</td>
</tr>
<tr>
<td>Rate of return</td>
<td>1%</td>
</tr>
</tbody>
</table>

The expected rate of return on SugarKane's stock is 6%, and its standard deviation is 14.73%. Thus SugarKane is almost as volatile as Best, yet its expected return is only a notch better than the T-bill rate. This cursory analysis makes SugarKane appear to be an unattractive investment. For Humanex, however, the stock holds great promise.

SugarKane offers excellent hedging potential for holders of Best stock because its return is highest precisely when Best's return is lowest—during a sugar crisis. Consider Humanex's portfolio when it splits its investment evenly between Best and SugarKane. The rate of return for each scenario is the simple average of the rates on Best and SugarKane because the portfolio is split evenly between the two stocks (see rule 3).

<table>
<thead>
<tr>
<th>Normal Year for Sugar</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullish Stock Market</td>
<td>Bearish Stock Market</td>
</tr>
<tr>
<td>Probability</td>
<td>.5</td>
</tr>
<tr>
<td>Rate of return</td>
<td>13.0%</td>
</tr>
</tbody>
</table>
The expected rate of return on Humeon’s hedged portfolio is 5.25% with a standard deviation of 4.83%.

Sily now summarizes the rewards and risk of the three alternatives:

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>All in Best Candy</td>
<td>10.50%</td>
<td>18.90%</td>
</tr>
<tr>
<td>Half in T-bills</td>
<td>7.75%</td>
<td>9.45%</td>
</tr>
<tr>
<td>Half in SugarKane</td>
<td>8.25%</td>
<td>4.83%</td>
</tr>
</tbody>
</table>

The numbers speak for themselves. The hedge portfolio with SugarKane clearly dominates the simple risk-reduction strategy of investing in safe T-bills. It has higher expected return and lower standard deviation than the one-half T-bill portfolio. The point is that, despite SugarKane’s large standard deviation of return, it is a hedge (risk reducer) for investors holding Best stock.

The risk of individual assets in a portfolio must be measured in the context of the effect of their return on overall portfolio variability. This example demonstrates that assets with returns that are inversely associated with the initial risky position are powerful hedge assets.

CONCEPT CHECK QUESTION

To quantify the hedging or diversification potential of an asset, we use the concepts of covariance and correlation. The covariance measures how much the returns on two risky assets move in tandem. A positive covariance means that asset returns move together. A negative covariance means that they vary inversely, as in the case of Best and SugarKane.

To measure covariance, we look at return "surprises," or deviations from expected value, in each scenario. Consider the product of each stock’s deviation from expected return in a particular scenario:

\[ (r_{\text{best}} - E(r_{\text{best}}))(r_{\text{sugar}} - E(r_{\text{sugar}})) \]

This product will be positive if the returns of the two stocks move together, that is, if both return exceed their expectations or both fall short of those expectations in the scenario in question. On the other hand, if one stock’s return exceeds its expected value while the other falls short, the product will be negative. Thus a good measure of the degree to which the returns move together is the expected value of this product across all scenarios, which is defined as the covariance:

\[ \text{Cov}(r_{\text{best}}, r_{\text{sugar}}) = \sum \text{Pr}(s)(r_{\text{best}}(s) - E(r_{\text{best}}))(r_{\text{sugar}}(s) - E(r_{\text{sugar}})) \]

(6.4)

In this example, with \( E(r_{\text{best}}) = 10.5\% \) and \( E(r_{\text{sugar}}) = 6\% \), and with returns in each scenario summarized in the next table, we compute the covariance by applying equation 6.4. The covariance between the two stocks is:

\[ \text{Cov}(r_{\text{best}}, r_{\text{sugar}}) = 5(25 - 10.5)(1 - 6) + 3(10 - 10.5)(-5 - 6) + 2(-25 - 10.5)35 - 6 = -240.5 \]

The negative covariance confirms the hedging quality of SugarKane stock relative to Best Candy. SugarKane’s returns move inversely with Best’s.
An easier statistic to interpret than the covariance is the correlation coefficient, which scales the covariance to a value between $-1$ (perfect negative correlation) and $+1$ (perfect positive correlation). The correlation coefficient between two variables equals their covariance divided by the product of the standard deviations. Denoting the correlation by the Greek letter $\rho$, we find that

$$\rho(\text{Best}, \text{SugarKane}) = \frac{\text{Cov}(r_{\text{Best}}, r_{\text{SugarKane}})}{\sigma_{\text{Best}} \sigma_{\text{SugarKane}}}$$

$$= \frac{-240.5}{18.9 \times 14.73} = -0.86$$

This large negative correlation (close to $-1$) confirms the strong tendency of Best and SugarKane stocks to move inversely, or "out of phase" with one another.

The impact of the covariance of asset returns on portfolio risk is apparent in the following formula for portfolio variance.

**Rule 5** When two risky assets with variances $\sigma_1^2$ and $\sigma_2^2$ respectively, are combined into a portfolio with portfolio weights $w_1$ and $w_2$, respectively, the portfolio variance $\sigma_p^2$ is given by

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \text{Cov}(r_1, r_2)$$

In this example, with equal weights in Best and SugarKane, $w_1 = w_2 = 0.5$, and with $\sigma_{\text{Best}} = 18.9\%$, $\sigma_{\text{SugarKane}} = 14.73\%$, and $\text{Cov}(r_{\text{Best}}, r_{\text{SugarKane}}) = -240.5$, we find that

$$\sigma_p^2 = (0.5^2 \times 18.9^2) + (0.5^2 \times 14.73^2) + [2 \times 0.5 \times 0.5 \times (-240.5)] = 23.3$$

so that $\sigma_p = \sqrt{23.3} = 4.83\%$. Precisely the same answer for the standard deviation of the returns on the hedged portfolio that we derived earlier from the scenario analysis.

Rule 5 for portfolio variance highlights the effect of covariance on portfolio risk. A positive covariance increases portfolio variance, and a negative covariance acts to reduce portfolio variance. This makes sense because returns on negatively correlated assets tend to be offsetting, which stabilizes portfolio returns.

Basically, hedging involves the purchase of a risky asset that is negatively correlated with the existing portfolio. This negative correlation makes the volatility of the hedge asset a risk-reducing feature. A hedge strategy is a powerful alternative to the simple risk-reduction strategy of including a risk-free asset in the portfolio.

In later chapters we will see that, in a rational market, hedge assets will offer relatively low expected rates of return. The perfect hedge, an insurance contract, is by design perfectly negatively correlated with a specified risk. As one would expect in a "no free lunch" world, the insurance premium reduces the portfolio's expected rate of return.
Suppose that the distribution of SugarKane stock were as follows:

<table>
<thead>
<tr>
<th>Bullish Stock Market</th>
<th>Bearish Stock Market</th>
<th>Sugar Crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>7%</td>
<td>-3%</td>
<td>20%</td>
</tr>
</tbody>
</table>

1. What would be its correlation with Gold?
2. Is SugarKane stock a useful hedge asset now?
3. Calculate the portfolio rate of return in each scenario and the standard deviation of the portfolio from the same to returns. Then evaluate each using rule 5.
4. Are the two methods of computing portfolio standard deviations consistent?

SUMMARY

1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero-risk premium. It will not be undertaken by a risk-averse investor.
3. Investors’ preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if it is received with certainty, would yield the same utility as the risky portfolio.
5. Hedging is the purchase of a risky asset to reduce the risk of a portfolio. The negative correlation between the hedge asset and the initial portfolio turns the volatility of the hedge asset into a risk-reducing feature. When a hedge asset is perfectly negatively correlated with the initial portfolio, it serves as a perfect hedge and works like an insurance contract on the portfolio.

KEY TERMS

- risk premium
- risk averse utility
- certainty equivalent rate
- risk neutral
- risk lover
- mean-variance (M-V)
- criterion
- indiffence curve
- hedging
- diversification
- expected return
- variance
- standard deviation
- covariance
- correlation
- coefficient

PROBLEMS

1. Consider a risky portfolio. The end-of-year cash flow derived from the portfolio will be either $70,000 or $200,000 with equal probabilities of .5. The alternative risk-free investment in T-bills pays 6% per year.
   a. If you require a risk premium of 8%, how much will you be willing to pay for the portfolio?
   b. Suppose that the portfolio can be purchased for the amount you found in (a). What will be the expected rate of return on the portfolio?
Consider historical data showing that the average annual rate of return on the S&P 500 portfolio over the past 70 years has averaged about 8.5% more than the Treasury bill return and that the S&P 500 standard deviation has been about 29% per year. Assume these values are representative of investors' expectations for future performance and that the current T-bill rate is 5%. Use these values to solve problems 10 to 12.

16. Calculate the expected return and variance of portfolios invested in T-bills and the S&P 500 index with weights as follows:

<table>
<thead>
<tr>
<th>W_T-bill</th>
<th>W_S&amp;P 500</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
</tr>
</tbody>
</table>

11. Calculate the utility levels of each portfolio of problem 10 for an investor with \( A = 3 \). What do you conclude?

12. Repeat problem 11 for an investor with \( A = 5 \). What do you conclude?

Reconsider the Best and SugarKane stock market hedging example in the text, but assume for questions 13 to 15 that the probability distribution of the rate of return on SugarKane stock is as follows:

<table>
<thead>
<tr>
<th>Bullish Stock Market</th>
<th>Bearish Stock Market</th>
<th>Sugar Crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Rate of return</td>
<td>10%</td>
<td>-5%</td>
</tr>
</tbody>
</table>

13. If Humanex's portfolio is half Best stock and half SugarKane, what are its expected return and standard deviation? Calculate the standard deviation from the portfolio returns in each scenario.

14. What is the covariance between Best and SugarKane?

15. Calculate the portfolio standard deviation using rule 5 and show that the result is consistent with your answer to question 13.

**Solutions to Concept Checks**

1. The expected rate of return on the risky portfolio is \( \$22,000/\$100,000 = 0.22 \), or 22%. The T-bill rate is 5%. The risk premium therefore is 22% - 5% = 17%.

2. The investor is taking on exchange rate risk by investing in a pound-denominated asset. If the exchange rate moves in the investor's favor, the investor will benefit and will earn more from the U.K. bill than the U.S. bill. For example, if both the U.S. and U.K. interest rates are 5%, and the current exchange rate is $1.50 per pound, a $1.50 investment today can buy one pound, which can be invested in England at a certain rate of 5%, for a year-end value of 1.05 pounds. If the year-end exchange rate is $1.60 per pound, the 1.05 pounds can be exchanged for 1.05 \times $1.60 = $1.68 for a rate of return in dollars of 1 + r = $1.68/$1.50 = 1.12, or 12%, more than is available from U.S. bills. Therefore, if the investor expects favorable exchange rate movements, the U.K. bill is a speculative investment. Otherwise, it is a gamble.
3. For the $A = 4$ investor the utility of the risky portfolio is

$$U = 20 - (0.005 \times 4 \times 20^2) = 12$$

while the utility of bills is

$$U = 7 - (0.005 \times 4 \times 0) = 7$$

The investor will prefer the risky portfolio to bills. (Of course, a mixture of bills and the portfolio might be even better, but that is not a choice here.) For the $A = 8$ investor, the utility of the risky portfolio is

$$U = 20 - (0.005 \times 8 \times 20^2) = 4$$

while the utility of bills is again $7$. The more risk-averse investor therefore prefers the risk-free alternative.

4. The less risk-averse investor has a shallower indifference curve. An increase in risk requires less increase in expected return to restore utility to the original level.

5. Despite the fact that gold investments in isolation seem dominated by the stock market, gold still might play a useful role in a diversified portfolio. Because gold and stock market returns have very low correlation, stock investors can reduce their portfolio risk by placing part of their portfolios in gold.

6. a. With the given distribution for SugarKane, the scenario analysis looks as follows:

<table>
<thead>
<tr>
<th></th>
<th>Normal Year for Sugar</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bullish Stock Market</td>
<td>Bearish Stock Market</td>
</tr>
<tr>
<td>Probability</td>
<td>.5</td>
<td>.3</td>
</tr>
<tr>
<td>Rate of Return (%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Best Candy</td>
<td>25</td>
<td>10</td>
</tr>
<tr>
<td>SugarKane</td>
<td>7</td>
<td>-5</td>
</tr>
<tr>
<td>T-bills</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
The expected return and standard deviation of Sugar Kane is now

\[ \text{E}(\text{Sugar Kane}) = (5 \times 7) + .3(-5) + .2(20) = 6 \]
\[ \sigma_{\text{Sugar Kane}} = \sqrt{(7 - 6)^2 + (-5 - 6)^2 + (20 - 6)^2} = 8.72 \]

The covariance between the returns of Best and Sugar Kane is

\[ \text{Cov}(\text{Sugar Kane, Best}) = .5(7 - 6)(25 - 10.5) + .3(-5 - 6)(10 - 10.5) + .2(20 - 6)(-25 - 10.5) = -90.5 \]

and the correlation coefficient is

\[ \rho(\text{Sugar Kane, Best}) = \frac{\text{Cov}(\text{Sugar Kane, Best})}{\sigma_{\text{Sugar Kane}} \cdot \sigma_{\text{Best}}} = \frac{-90.5}{8.72 \times 18.90} = - .55 \]

The correlation is negative, but less than before \((- .55 \text{ instead of } - .86)\) so we expect that Sugar Kane will now be a less powerful hedge than before. Investing 50% in Sugar Kane and 50% in Best will result in a portfolio probability distribution of

<table>
<thead>
<tr>
<th>Probability</th>
<th>5</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio return</td>
<td>16</td>
<td>2.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

resulting in a mean and standard deviation of

\[ \text{E}(\text{Hedged Portfolio}) = (5 \times 16) + (3 \times 2.5) + (2 \times -2.5) = 8.25 \]
\[ \sigma_{\text{Hedged Portfolio}} = \sqrt{(16 - 8.25)^2 + (2.5 - 8.25)^2 + (-2.5 - 8.25)^2} = 7.94 \]

b. It is obvious that even under these circumstances, the hedging strategy dominates the risk-reducing strategy that uses T-bills (which results in \(\text{E}(\sigma) = 7.75\% \), \(\sigma = 9.45\%\)). At the same time, the standard deviation of the hedged position (7.94%) is not as low as it was using the original data.

c, d. Using rule 5 for portfolio variance, we would find that

\[ \sigma^2 = (.5^2 \times \sigma_{\text{best}}^2) + (.5^2 \times \sigma_{\text{Keane}}^2) + [2 \times .5 \times .5 \times \text{Cov}(\text{Sugar Kane, Best})] \]
\[ = (.5^2 \times 18.9^2) + (.5^2 \times 8.72^2) + [2 \times .5 \times .5 \times (-90.5)] = 63.06 \]

which implies that \(\sigma = 7.94\%\), precisely the same result that we obtained by analyzing the scenarios directly.

APPENDIX A: A DEFENSE OF MEAN-VARIANCE ANALYSIS

Describing Probability Distributions

The axiom of risk aversion needs little defense. So far, however, our treatment of risk has been limiting in that it took the variance (or, equivalently, the standard deviation) of portfolio returns as an adequate risk measure. In situations in which variance alone is not adequate to measure risk, this assumption is potentially restrictive. Here we provide some justification for mean-variance analysis.