Chapter 8

Inferences Based on a Single Sample: Tests of Hypothesis

8.1 The Elements of a Test of Hypothesis

The objective of a statistical test is to test an hypothesis concerning the value of one or more population parameters. A statistical test involves the following four elements:

1. **Null and alternative hypotheses**: The purpose of hypothesis testing is to test the viability of the null hypothesis in the light of experimental data. The null hypothesis is the hypothesis about a population parameter. The alternative hypothesis is the hypothesis that the researchers wish to support. The null hypothesis is a contradiction of the alternative hypothesis. The null and alternative hypotheses are denoted by $H_0$ and $H_a$ respectively.

**Extra Example 1**: Suppose we want to know whether the average income of the FIU students is $20,000. Then we have the following two-sided alternative.

\[
H_0 : \mu = 20,000 \\
H_a : \mu \neq 20,000
\]

**Extra Example 2**: Suppose we want to know whether the average age of the FIU students exceeds 25 years (which is equivalent to say, not more than 25). Then we have the following one-sided (upper or right tailed test) alternative.

\[
H_0 : \mu \leq 25 \\
H_a : \mu > 25
\]
Suppose we want to know whether the average age of the FIU students is less than 25 years (which is equivalent to say, at least 25). Then we have the following **one-sided** (lower or left tailed test) alternative.

\[
H_0 : \mu \geq 25 \\
H_a : \mu < 25
\]

2. **Test statistic:** A statistic computed from sample measurements that will be used to test the null hypothesis. OR, The sample quantity on which the decision to support \(H_0\) or \(H_a\) is based is called the test statistic.

3. **Critical region or rejection region:** The set of values of the test statistic that leads to rejection of the null hypothesis in favor of the alternative hypothesis is called the critical or rejection region.

4. **Decision Rules:** Reject \(H_0\) at \(\alpha\)% level of significance, if the calculated value of the test statistic (observed value) falls in the rejection region.

There are four possible outcomes of a test of hypothesis presented in Table 8.1.

<table>
<thead>
<tr>
<th>Truth about the population</th>
<th>Decision based on sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H_0) is true</td>
<td>Type I</td>
</tr>
<tr>
<td>(H_0) is false</td>
<td>Correct Decision</td>
</tr>
</tbody>
</table>

**Type I and Type II Errors**

Two kinds of errors are committed when testing the hypotheses or making decision about the null hypothesis.

**Type I Error:** If \(H_0\) is rejected when it is true (ie, rejecting a true null hypothesis). That is, a true null hypothesis can be incorrectly rejected.

**Type II Error:** If \(H_0\) is fail to reject when it is false (ie, accepting a false null hypothesis). That is a false null hypothesis can fail to be rejected.

The goodness of statistical test of hypothesis is measured by the probabilities of making a type I or a type II error, denoted by \(\alpha\) and \(\beta\) respectively.

**Level of Significance:** In hypothesis testing, the significance level is the criterion used for rejecting the null hypothesis, which measures the probability of type II error. That is

\[
\alpha = P(\text{Type I error}) = P(\text{reject } H_0|H_0 \text{ is true})
\]
Here, $\alpha$ is also called the **level of significance** and $(1 - \alpha)$ is called the **confidence coefficient**.

**Power of the test:**

$$\beta = P(\text{Type II error}) = P(\text{do not reject } H_0|H_1 \text{ is true})$$

Power=$(1 - \beta) = p(\text{reject } H_0|H_0 \text{ is false})=P(\text{rejecting a false null hypothesis}).$

**Note:** We can not minimize both type I and type II errors at the same time. Therefore, for a given $\alpha$, one would like to minimize the type II error, *i.e.*, maximize the power of the test.

Some selected significance levels and the corresponding critical values are presented in Table 8.2.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$z_{\alpha}$ (two tail)</th>
<th>$z_{\alpha}$ (One tail)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>1.645</td>
<td>1.28</td>
</tr>
<tr>
<td>0.05</td>
<td>1.96</td>
<td>1.645</td>
</tr>
<tr>
<td>0.02</td>
<td>2.33</td>
<td>2.05</td>
</tr>
<tr>
<td>0.01</td>
<td>2.58</td>
<td>2.33</td>
</tr>
</tbody>
</table>

**Construction of a test statistic for testing population mean or proportion**

$$Test \ statistic = \frac{Estimator - \text{Null Value}}{SE(Estimator)} \quad (8.1)$$

### 8.2 Formulating Hypotheses and Setting Up the Rejection Region

**Steps for Selecting the Null and Alternative Hypotheses**

1. Select the alternative hypothesis as that which the sampling experiment in intended to establish. The alternative hypothesis will assume one of the following three forms:
   a. One tailed, upper (right) tailed (say, $H_a: \mu > \mu_0$)
   b. One tailed, lower (left) tailed (say, $H_a: \mu < \mu_0$)
   c. Two tailed, upper (right) tailed (say, $H_a: \mu \neq \mu_0$)

2. Select the null hypothesis
   $H_0: \mu = \mu_0$.

**One-tailed test:** A one-tailed test of hypothesis is one in which the alternative hypothesis is directional and includes the symbol “<” or “>”. 
Two-tailed test: A two-tailed test of hypothesis is one in which the alternative hypothesis does not specify departure from $H_0$ in a particular direction and is written with the symbol “$\neq$”.

Example 8.1, page 332
Example 8.2, page 333

8.3 Test of Hypotheses about a Population Mean: Normal (z) Statistic

Assumptions:

1. The $n$ observations (measurements) in the sample were randomly selected from the population.
2. Sample size is large, i.e., $n \geq 30$.

The test of hypotheses about the population means are summarized in Table 8.3.

<table>
<thead>
<tr>
<th>$H_0$</th>
<th>$H_a$</th>
<th>Test Statistic</th>
<th>Rejection Region</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = \mu_0$</td>
<td>$\mu \neq \mu_0$</td>
<td>$z_0 = \frac{\sqrt{n} (\bar{x} - \mu_0)}{\sigma}$</td>
<td>$</td>
<td>z_0</td>
</tr>
<tr>
<td>$\mu \leq \mu_0$</td>
<td>$\mu &gt; \mu_0$</td>
<td>$z_0 = \frac{\sqrt{n} (\bar{x} - \mu_0)}{\sigma}$</td>
<td>$z_0 &gt; z_{\alpha}$</td>
<td>$P(z &gt; z_0)$</td>
</tr>
<tr>
<td>$\mu \geq \mu_0$</td>
<td>$\mu &lt; \mu_0$</td>
<td>$z_0 = \frac{\sqrt{n} (\bar{x} - \mu_0)}{\sigma}$</td>
<td>$z_0 &lt; -z_{\alpha}$</td>
<td>$P(z &lt; z_0)$</td>
</tr>
</tbody>
</table>

Note: From table 8.3 we observed that the equality signs occur with the null hypothesis ($H_0$) only. However, not equal, strictly less than or strictly greater than signs occur with alternative hypothesis ($H_a$). From the wording of the questions, you will be able to define both null and alternative hypotheses. If the population standard deviation, $\sigma$ is unknown, then replace it with $s$, the sample standard deviation.

Extra Example 3: A researcher wishes to test the claim that the average age of lifeguards in Ocean City is greater than 24 years. She selects a random sample of 51 guards and finds the mean of the sample as 24.7 years, with a standard deviation of 2 years. Do the sample data gives the sufficient evidence to reject the researcher’s claim?

(a) Write your null and alternative hypotheses.

(b) Find the value of the test statistic for testing hypothesis in (a).
(c) Write down your decision about the null hypothesis at $\alpha = 0.01$ level.

Example 8.4, page 338.


8.4 Observed Significance Level: $p$-Values

Definition 8.1 P-value: The P-value is the probability that the test statistic will take on a value that is at least as extreme as the observed value of the test statistic when the null hypothesis is true. The P-value is also called the observed significance level of the test.

Note: P-value = observed level of significance = $\hat{\alpha}$.

Decision: Smaller the P-value, the more likely to reject the null hypothesis. More specifically, one would reject the null hypothesis at $\alpha$ level of significance, if

$$P-value \leq \alpha$$

Extra Example 4: Suppose the P-value of a test statistic is found to be 0.03. We are testing the null hypothesis at 5% level of significance. Do we accept or reject the null hypothesis?

Answer: We do reject the null hypothesis at 5% level of significance, since the P-value is less than 0.05.

Extra Example 5: Suppose the P-value of a test statistic is found to be 0.13. We are testing the null hypothesis at 10% level of significance. Do we accept or reject the null hypothesis?

Answer: We do not reject the null hypothesis at 10% level of significance, since the P-value is greater than 0.10.

Example 8.5, page 344.

8.5 Test of Hypotheses about a Population Mean: Student’s $t$-Statistic

Assumptions:

1. The $n$ observations (measurements) in the sample were randomly selected from the population.
2. The parent population is normal.
The test of hypotheses about the population means are summarized in Table 8.4.

### Table 8.4: Small-Sample Test of Hypothesis about $\mu$

<table>
<thead>
<tr>
<th>$H_0$</th>
<th>$H_a$</th>
<th>Test Statistic</th>
<th>Rejection Region</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = \mu_0$</td>
<td>$\mu \neq \mu_0$</td>
<td>$t_0 = \frac{\sqrt{n} (\bar{x} - \mu_0)}{s}$</td>
<td>$</td>
<td>t_0</td>
</tr>
<tr>
<td>$\mu \leq \mu_0$</td>
<td>$\mu &gt; \mu_0$</td>
<td>$t_0 = \frac{\sqrt{n} (x - \mu_0)}{s}$</td>
<td>$t_0 &gt; t_\alpha$</td>
<td>$P(t &gt; t_0)$</td>
</tr>
<tr>
<td>$\mu \geq \mu_0$</td>
<td>$\mu &lt; \mu_0$</td>
<td>$t_0 = \frac{\sqrt{n} (x - \mu_0)}{s}$</td>
<td>$t_0 &lt; -t_\alpha$</td>
<td>$P(t &lt; t_0)$</td>
</tr>
</tbody>
</table>

Example 8.7, page 350.

Example 8.8, page 351.

**Extra Example 6:** A chemical plant is required to maintain ambient sulfur levels in the working environment atmosphere at an average level of no more than 12.50. The results of 15 randomly timed measurements of the sulfur level produced a sample mean of $\bar{x} = 15.82$ and a sample standard deviation of $s = 2.82$. Test the hypothesis that the true average sulfur level is not more than 12.50. Write your decision about the null hypothesis. Use $\alpha = 0.05$.

### 8.6 Large-Sample Test of Hypothesis about a Population Proportion

**Assumptions:**

1. The $n$ observations (measurements) in the sample were randomly selected from a binomial population.
2. Sample size is large, i.e., $n \geq 30$.

The test of hypotheses about the population proportions are summarized in Table 8.5.
Table 8.5: Large-Sample Test of Hypothesis about $p$

<table>
<thead>
<tr>
<th>$H_0$</th>
<th>$H_a$</th>
<th>Test statistic</th>
<th>Rejection region</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = p_0$</td>
<td>$p \neq p_0$</td>
<td>$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$</td>
<td>$</td>
<td>z_0</td>
</tr>
<tr>
<td>$p \leq p_0$</td>
<td>$p &gt; p_0$</td>
<td>$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$</td>
<td>$z_0 &gt; z_{\alpha}$</td>
<td>$P(z &gt; z_0)$</td>
</tr>
<tr>
<td>$p \geq p_0$</td>
<td>$p &lt; p_0$</td>
<td>$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$</td>
<td>$z_0 &lt; -z_{\alpha}$</td>
<td>$P(z &lt; z_0)$</td>
</tr>
</tbody>
</table>

where $\hat{p} = \frac{x}{n}$

Example 8.9, page 357.

Example 8.10, page 358.

Exercise 8.64, page 360.

Extra Example 7: A researcher claims that less than 5% of the adult people in London, Ontario do not have any car. In an experiment of this claim, a random sample of 200 adult people were randomly chosen from London city and found that 8 of them do not have any car. Do these data provide sufficient evidence that less than 5% of the adult people in Vancouver do not have any car?

(a) Set up the null and the alternative hypotheses.

(b) Calculate the value of the test statistic.

(c) Calculate the P-value of the test statistic in (b) and write your decision about the null hypothesis at $\alpha = 0.10$.

Extra Example 8: In a random sample of 200 college students, 45 had taken Statistics in high school. Using $\alpha = 0.04$, do you have enough evidence to conclude the proportion of college students who have not taken Statistics in high school differs from 0.75?