1. a. A production function has constant returns to scale if increasing all factors of production by an equal percentage causes output to increase by the same percentage. Mathematically, a production function has constant returns to scale if \( zY = F(zK, zL) \) for any positive number \( z \). That is, if we multiply both the amount of capital and the amount of labor by some amount \( z \), then the amount of output is multiplied by \( z \). For example, if we double the amounts of capital and labor we use (setting \( z = 2 \)), then output also doubles.

   To see if the production function \( Y = F(K, L) = K^{1/2}L^{1/2} \) has constant returns to scale, we write:

   \[
   F(zK, zL) = (zK)^{1/2}(zL)^{1/2} = zK^{1/2}L^{1/2} = zY.
   \]

   Therefore, the production function \( Y = K^{1/2}L^{1/2} \) has constant returns to scale.

b. To find the per-worker production function, divide the production function \( Y = K^{1/2}L^{1/2} \) by \( L \):

   \[
   \frac{Y}{L} = \frac{K^{1/2}L^{1/2}}{L}.
   \]

   If we define \( y = Y/L \), we can rewrite the above expression as:

   \[
   y = K^{1/2}/L^{1/2}.
   \]

   Defining \( k = K/L \), we can rewrite the above expression as:

   \[
   y = k^{1/2}.
   \]

c. We know the following facts about countries A and B:

   \( \delta = \) depreciation rate = 0.05,
   \[ s_a = \text{saving rate of country A} = 0.1, \]
   \[ s_b = \text{saving rate of country B} = 0.2, \]

   and \( y = k^{1/2} \) is the per-worker production function derived in part (b) for countries A and B.

   The growth of the capital stock \( \Delta k \) equals the amount of investment \( s/f(k) \), less the amount of depreciation \( \delta k \). That is, \( \Delta k = s/f(k) - \delta k \). In steady state, the capital stock does not grow, so we can write this as \( s/f(k) = \delta k \).

   To find the steady-state level of capital per worker, plug the per-worker production function into the steady-state investment condition, and solve for \( k^* \):

   \[
   s\left(k^{1/2}\right) = \delta k.
   \]

   Rewriting this:

   \[
   k^{1/2} = \frac{s}{\delta} \]

   \[
   k = \left(\frac{s}{\delta}\right)^2.
   \]

   To find the steady-state level of capital per worker \( k^* \), plug the saving rate for each country into the above formula:

   Country A: \( k_a^* = (s_a/\delta)^2 = (0.1/0.05)^2 = 4. \)

   Country B: \( k_b^* = (s_b/\delta)^2 = (0.2/0.05)^2 = 16. \)

   Now that we have found \( k^* \) for each country, we can calculate the steady-state levels of income per worker for countries A and B because we know that \( y = k^{1/2} \).
\[ y_a^* = (4)^{1/2} = 2. \]
\[ y_b^* = (16)^{1/2} = 4. \]

We know that out of each dollar of income, workers save a fraction \( s \) and consume a fraction \((1 - s)\). That is, the consumption function is \( c = (1 - s)y \). Since we know the steady-state levels of income in the two countries, we find

Country A: \[ c_a^* = (1 - s_a)y_a^* = (1 - 0.1)(2) = 1.8. \]

Country B: \[ c_b^* = (1 - s_b)y_b^* = (1 - 0.2)(4) = 3.2. \]

d. Using the following facts and equations, we calculate income per worker \( y \), consumption per worker \( c \), and capital per worker \( k \):

\[ s_a = 0.1. \]
\[ s_b = 0.2. \]
\[ \delta = 0.05. \]
\[ k_c = 2 \text{ for both countries}. \]
\[ y = k^{1/2}. \]
\[ c = (1 - s)y. \]

<table>
<thead>
<tr>
<th>Country A</th>
<th>Year</th>
<th>( k )</th>
<th>( y = k^{1/2} )</th>
<th>( c = (1 - s_a)y )</th>
<th>( i = s_a y )</th>
<th>( \Delta k )</th>
<th>( \Delta k = i - \Delta k )</th>
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<tbody>
<tr>
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<td>1.273</td>
<td>0.141</td>
<td>0.100</td>
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<tr>
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<tr>
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<td>1.323</td>
<td>0.147</td>
<td>0.108</td>
<td>0.039</td>
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<table>
<thead>
<tr>
<th>Country B</th>
<th>Year</th>
<th>( k )</th>
<th>( y = k^{1/2} )</th>
<th>( c = (1 - s_b)y )</th>
<th>( i = s_b y )</th>
<th>( \Delta k )</th>
<th>( \Delta k = i - \Delta k )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.138</td>
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</tr>
</tbody>
</table>

Note that it will take five years before consumption in country B is higher than consumption in country A.
2. a. The production function in the Solow growth model is \( Y = F(K, L) \), or expressed in terms of output per worker, \( y = f(k) \). If a war reduces the labor force through casualties, then \( L \) falls but \( k = K/L \) rises. The production function tells us that total output falls because there are fewer workers. Output per worker increases, however, since each worker has more capital.

b. The reduction in the labor force means that the capital stock per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has a capital stock that is higher than the steady-state level. This is shown in Figure 7–2 as an increase in capital per worker from \( k^* \) to \( k_1 \). As the economy returns to the steady state, the capital stock per worker falls from \( k_1 \) back to \( k^* \), so output per worker also falls.

Hence, in the transition to the new steady state, the growth of output per worker is slower than normal. In the steady state, we know that the growth rate of output per worker is equal to zero, given there is no technological change in this model. Therefore, in this case, the growth rate of output per worker must be less than zero until the new steady state is reached.

3. Suppose the economy begins with an initial steady-state capital stock below the Golden Rule level. The immediate effect of devoting a larger share of national output to investment is that the economy devotes a smaller share to consumption; that is, "living standards" as measured by consumption fall. The higher investment rate means that the capital stock increases more quickly, so the growth rates of output and output per worker rise. The productivity of workers is the average amount produced by each worker—that is, output per worker. So productivity growth rises. Hence, the immediate effect is that living standards fall but productivity growth rises.

In the new steady state, output grows at rate \( n \), while output per worker grows at rate zero. This means that in the steady state, productivity growth is independent of the rate of investment. Since we begin with an initial steady-state capital stock below the Golden Rule level, the higher investment rate means that the new steady state has a higher level of consumption, so living standards are higher.
Thus, an increase in the investment rate increases the productivity growth rate in the short run but has no effect in the long run. Living standards, on the other hand, fall immediately and only rise over time. That is, the quotation emphasizes growth, but not the sacrifice required to achieve it.

First, consider steady states. In Figure 7–3, the slower population growth rate shifts the line representing population growth and depreciation downward. The new steady state has a higher level of capital per worker, \( k^*_2 \), and hence a higher level of output per worker.

![Figure 7–3](image)

What about steady-state growth rates? In steady state, total output grows at rate \( n \), whereas output per-worker grows at rate \( \delta \). Hence, slower population growth will lower total output growth, but per-worker output growth will be the same.

Now consider the transition. We know that the steady-state level of output per worker is higher with low population growth. Hence, during the transition to the new steady state, output per worker must grow at a rate faster than \( \delta \) for a while. In the decades after the fall in population growth, growth in total output will transition to its new lower level while growth in output per worker will jump up but then transition back to zero.

5.

a. To find output per worker \( y \) we divide total output by the number of workers:

\[
\frac{Y}{L} = \frac{K^a[(1-u)L]^\alpha}{L}
\]

\[
y = \left( \frac{K}{L} \right)^a (1-u)^{\alpha-a}
\]

\[
y = k^a(1-u)^{1-\alpha}
\]
where the final step uses the definition $k = \frac{K}{L}$. Notice that unemployment reduces the amount of output per worker for any given capital–labor ratio because some of the workers are not producing anything.

The steady state is the level of capital per worker at which the increase in capital per worker from investment equals its decrease from depreciation and population growth (see Chapter 7 for more details).

\[ sy = (\delta + n)k \]
\[ sk^\alpha (1 - u)^{1-\alpha} = (\delta + n)k \]
\[ k^* = (1 - u) \left( \frac{s}{\delta + n} \right)^{\frac{1}{1-\alpha}} \]

Unemployment lowers the marginal product of capital per worker and, hence, acts like a negative technological shock that reduces the amount of capital the economy can maintain in steady state. Figure 7–4 shows this graphically: an increase in unemployment lowers the $s/k$ line and the steady-state level of capital per worker.

![Figure 7-4](image)

Finally, to get steady-state output per worker, plug the steady-state level of capital per worker into the production function:
\[ y^* = \left(1 - u^*\right) \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-a}} \left(1 - u^*\right)^{1-a} \]

\[ = \left(1 - u^*\right) \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-a}} \]

Unemployment lowers steady-state output for two reasons: for a given $k$, unemployment lowers $y$, and unemployment also lowers the steady-state value $k^*$.

b. Figure 7–5 below shows the pattern of output over time. As soon as unemployment falls from $u_1$ to $u_2$, output jumps up from its initial steady-state value of $y^*(u_1)$. The economy has the same amount of capital (since it takes time to adjust the capital stock), but this capital is combined with more workers. At that moment the economy is out of steady state: it has less capital than it wants to match the increased number of workers in the economy. The economy begins its transition by accumulating more capital, raising output even further than the original jump. Eventually the capital stock and output converge to their new, higher steady-state levels.