FOURIER SERIES PART II: CONVERGENCE

We have seen in the previous note how to associate to a $2\pi$-periodic function $f$ a Fourier series
\[ a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \]

Now we are going to investigate how the Fourier series represents $f$. Let us first introduce the following notation. For $N = 0, 1, 2, \cdots$, we denote by $S_N f(x)$ the $N$-th partial sum of the Fourier series of $f$. That is,
\[ S_N f(x) = a_0/2 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx). \]

Hence
\[ S_0 f(x) = a_0/2; \]
\[ S_1 f(x) = a_0/2 + a_1 \cos x + b_1 \sin x; \]
\[ S_2 f(x) = a_0/2 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \]
\[ \vdots \]

The infinite series is therefore $\lim_{N \to \infty} S_N f$. The Fourier series converges at a point $x$ if $\lim_{N \to \infty} S_N f(x)$ exists.

We consider the functions and their Fourier series of examples 1, 2, and 3 of the previous note and see how the graphs of partial sums $S_N f$ compare to those of $f$.

1. EXAMPLES

Example 1. For $f(x) = |x|$ on $[-\pi, \pi]$, we found
\[ |x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2}. \]

Thus,
\[ S_0 f(x) = \frac{\pi}{2}; \]
\[ S_1 f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x; \]
\[ S_2 f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x - \frac{4}{9\pi} \cos 3x; \]
\[ S_3 f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x - \frac{4}{9\pi} \cos 3x - \frac{4}{25\pi} \cos 5x. \]

It appears that as $N$ gets larger, the graph of $S_N f$ gets closer to that of $f$.  

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Example 2. For the $2\pi$-periodic function $f$ of example 2 defined by

\[ f(x) = \begin{cases} 
1 & \text{if } 0 < x < \pi; \\
-1 & \text{if } -\pi < x < 0 
\end{cases} \]

we found the Fourier series

\[ f(x) \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)}. \]

Thus,

\[ S_1 f(x) = \frac{4 \sin x}{\pi}; \]
\[ S_3 f(x) = \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi}; \]
\[ S_5 f(x) = \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi} + \frac{4 \sin 5x}{5\pi}; \]
\[ S_7 f(x) = \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi} + \frac{4 \sin 5x}{5\pi} + \frac{4 \sin 7x}{7\pi}. \]

Again it appears that as $N$ increases $S_N f$ gets closer to $f$ at the points where $f$ is continuous.

Example 3. For the $2\pi$-periodic function $f$ of example 3 defined by

\[ f(x) = \begin{cases} 
x & \text{if } 0 < x < \pi; \\
0 & \text{if } -\pi < x < 0 
\end{cases} \]

with Fourier series

\[ \frac{\pi}{4} - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \sin jx}{j} \].
The first partial sums are

\[ S_0 f(x) = \frac{\pi}{4} \]
\[ S_1 f(x) = \frac{\pi}{4} - \frac{2 \cos \pi x}{\pi} + \sin x \]
\[ S_2 f(x) = \frac{\pi}{4} - \frac{2 \cos \pi x}{\pi} + \sin x - \frac{\sin 2x}{2} \]
\[ S_3 f(x) = \frac{\pi}{4} - \frac{2 \cos \pi x}{\pi} + \sin x - \frac{\sin 2x}{2} - \frac{2 \cos 3x}{3} + \sin 3x \]

2. Pointwise Convergence of Fourier series

The above examples suggest that the $N$-th partial sums $S_N f$ converge to $f$. This is indeed the case at each point where $f$ is continuous. At each discontinuity, the partial sums approach the average value of $f$. To be precise, we define the average of $f$ at a point $x_0$ as

\[ f_{av}(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{1}{2} \left( \lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right) \]

Hence if $f$ is continuous at $x_0$, then $f_{av}(x_0) = f(x_0)$. For example for the $2\pi$-periodic function $f$ of example 3 defined by $f(x) = \begin{cases} x & \text{if } 0 < x < \pi; \\ 0 & \text{if } -\pi < x < 0. \end{cases}$ we
have $f_{av}(x) = f(x)$ for $x \neq (2k + 1)\pi$ (with $k \in \mathbb{Z}$) and

$$f_{av}((2k + 1)\pi) = \frac{f((2k + 1)\pi^+) + f((2k + 1)\pi^-)}{2} = \frac{\pi}{2} \quad k = \pm 1, \pm 2, \pm 3, \cdots$$

The graph of $f_{av}$ is the following.

We have the following theorem.

**Theorem** (Pointwise convergence) Let $f \in C^1_p(\mathbb{R})$ be $2\pi$-periodic. Then the Fourier series of $f$ converges to $f_{av}$ at each point of $\mathbb{R}$. That is,

$$f_{av}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$
where
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad \text{and} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \left\{ \cos nx \right\} dx \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \left\{ \sin nx \right\} dx \]

Again this means that at all points \( x \) where \( f \) is continuous, we have
\[ f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx, \]
and at the points \( x_0 \) where \( f \) is discontinuous we have
\[ \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx_0 + b_n \sin nx_0. \]

To prove this theorem, we will need two lemmas

**Lemma 1.** (Riemann-Lebesgue Lemma) If \( f \) is piecewise smooth on an interval \([a, b]\), then
\[ \lim_{r \to \infty} \int_{a}^{b} f(x) \cos(rx) dx = 0 \]
\[ \lim_{r \to \infty} \int_{a}^{b} f(x) \sin(rx) dx = 0 \]

**Proof.** Since \( f \) is piecewise smooth, then there are finitely many points
\[ c_0 = a < c_1 < c_2 < \cdots < c_{n-1} < b = c_n \]
such that both \( f \) and its derivative \( f' \) are continuous in each interval \((c_{j-1}, c_j)\) \((j = 1, \cdots, n)\). Furthermore, \( f(c_k^±) \) and \( f'(c_k^±) \) are finite. Thus the integrals of \( f \) and \( f' \) exist in each subinterval. We have,
\[ \int_{a}^{b} f(x) \cos(rx) dx = \int_{c_0}^{c_1} f(x) \cos(rx) dx + \cdots + \int_{c_{n-1}}^{c_n} f(x) \cos(rx) dx \]
\[ = \sum_{j=1}^{n} \int_{c_{j-1}}^{c_j} f(x) \cos(rx) dx \]

We use integration by parts in each subinterval \([c_{j-1}, c_j]\) to obtain
\[ \int_{c_{j-1}}^{c_j} f(x) \cos(rx) dx = \left( \frac{f(x) \sin(rx)}{r} \right)_{c_{j-1}}^{c_j} - \int_{c_{j-1}}^{c_j} f'(x) \frac{\sin(rx)}{r} dx \]
\( (\text{we are assuming that } r > 0) \). Let \( M > 0 \) such that
\[ \sup_{a < x < b} |f(x)| < M \quad \text{and} \quad \sup_{a < x < b} |f'(x)| < M. \]

Then
\[ \left| \left( \frac{f(x) \sin(rx)}{r} \right)_{c_{j-1}}^{c_j} \right| \leq \left| f(c_j) \sin(rc_j) \right| + \left| f(c_{j-1}) \sin(rc_{j-1}) \right| \leq \frac{2M}{r} \]
and
\[ \left| \int_{c_{j-1}}^{c_j} f'(x) \frac{\sin(rx)}{r} dx \right| \leq \int_{c_{j-1}}^{c_j} \left| f'(x) \frac{\sin(rx)}{r} \right| dx \leq \frac{M(c_j - c_{j-1})}{r}. \]
It follows that
\[ \int_a^b f(x) \cos(rx) \, dx \leq \sum_{j=1}^n \left( \frac{2M}{r} + \frac{M(c_j - c_{j-1})}{r} \right) \leq \frac{2Mn + (b-a)}{r}. \]
Since \( \frac{2Mn + (b-a)}{r} \to 0 \) as \( r \to \infty \), then
\[ \lim_{r \to \infty} \int_a^b f(x) \cos(rx) \, dx = 0. \]
A similar argument gives the second limit of the lemma.

**Lemma 2.** for every \( x \in \mathbb{R}, x \neq 2k\pi \) with \( k \in \mathbb{Z} \), we have the identity
\[ \frac{1}{2} + \cos x + \cos(2x) + \cdots + \cos(Nx) = \frac{\sin((N + \frac{1}{2})x)}{2\sin \frac{x}{2}}. \]

**Proof.** Set \( T = \frac{1}{2} + \cos x + \cos(2x) + \cdots + \cos(Nx) \). By using \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \), we can rewrite \( T \) as
\[ T = \frac{1}{2} + \sum_{j=1}^N \frac{e^{ijx} + e^{-ijx}}{2} = \frac{1}{2} \left( 1 + \sum_{j=1}^N e^{ijx} + \sum_{j=1}^N e^{-ijx} \right) \]
Note that \( \sum_{j=1}^N e^{ijx} \) and \( \sum_{j=1}^N e^{-ijx} \) are geometric sums. The first with ratio \( e^{ix} \) and the second with ratio \( e^{-ix} \). Since \( x \neq 2k\pi \) these ratios are different from 1 and
\[ \sum_{j=1}^N e^{ijx} = \frac{e^{ix}(1 - e^{NX})}{1 - e^{ix}} = \frac{e^{ix} - e^{i(N+1)x}}{1 - e^{ix}} \]
and
\[ \sum_{j=1}^N e^{-ijx} = \frac{e^{-ix}(1 - e^{-iNX})}{1 - e^{-ix}} = \frac{e^{-ix} - e^{-i(N+1)x}}{1 - e^{-ix}}. \]
After reducing to the same denominator, the expression for \( T \) becomes
\[ T = \frac{e^{iNx} + e^{-iNx} - e^{i(N+1)x} - e^{-i(N+1)x}}{2(2 - (e^{ix} + e^{-ix}))} = \frac{\cos Nx - \cos(N+1)x}{2(1 - \cos x)}. \]
Now use the trigonometric identities
\[ \cos Nx = \cos((N + \frac{1}{2})x - x) = \cos((N + \frac{1}{2})x) \cos \frac{x}{2} + \sin((N + \frac{1}{2})x) \sin \frac{x}{2} \]
\[ \cos(N+1)x = \cos((N + \frac{1}{2})x + \frac{x}{2}) = \cos((N + \frac{1}{2})x) \cos \frac{x}{2} - \sin((N + \frac{1}{2})x) \sin \frac{x}{2}. \]
Hence,
\[ \cos Nx - \cos(N+1)x = 2 \sin((N + \frac{1}{2})x) \sin \frac{x}{2}. \]
We also have
\[ 1 - \cos x = 2 \sin^2 \frac{x}{2}. \]
Therefore
\[ T = \frac{2 \sin((N + \frac{1}{2})x) \sin \frac{x}{2}}{4 \sin^2 \frac{x}{2}} = \frac{\sin(N + \frac{1}{2})x}{2 \sin \frac{x}{2}}. \]

After these two lemmas, we start the proof of the convergence of Fourier series. Let \( S_N f \) be the \( N \)-th partial sum of the Fourier series of \( f \). That is,
\[ S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx). \]

We would like to prove that
\[ \lim_{N \to \infty} S_N f(x) = f_{av}(x) = \frac{f(x^+) + f(x^-)}{2}. \]

We are going to use Lemma 2 and the definition of the Fourier coefficients \( a_j \) and \( b_j \) to rewrite \( S_N f \) in an integral form. Recall that
\[ a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos jt \, dt, \quad j = 0, 1, 2, \ldots \]
\[ b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin jt \, dt, \quad j = 1, 2, 3, \ldots \]

We can rewrite \( S_N f \) as
\[ S_N f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos jt \, dt + \sum_{j=1}^{N} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos jt \, dt \cos jx + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin jt \, dt \sin jx \right) \]
\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2} + \sum_{j=1}^{N} (\cos jt \cos jx + \sin jt \sin jx) \right) \, dt \]

The trigonometric identity \( \cos jt \cos jx + \sin jt \sin jx = \cos j(t - x) \) gives
\[ S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2} + \sum_{j=1}^{N} \cos j(t - x) \right) \, dt \]

Now Lemma 2 can be used to obtain
\[ S_N(x) = \int_{-\pi}^{\pi} f(t) \frac{\sin \left[ \left( N + \frac{1}{2} \right)(t - x) \right]}{2\pi \sin \left[ \frac{t - x}{2} \right]} \, dt \]

Define the function \( D_N(s) \), called the Dirichlet kernel, by
\[ D_N(s) = \begin{cases} \frac{\sin(N + \frac{1}{2})s}{2\pi \sin \frac{s}{2}} & \text{if } s \neq 2k\pi, \, k \in \mathbb{Z}; \\ \frac{2N + 1}{2\pi} & \text{if } s = 2k\pi, \, k \in \mathbb{Z}. \end{cases} \]

Note that \( D_N : \mathbb{R} \to \mathbb{R} \) is an even and continuous function. That \( D_N \) is contin-
uous at a point $s_0 = 2k\pi$ follows from L'Hopital’s rule:

$$\lim_{s \to 2k\pi} D_N(s) = \lim_{s \to 2k\pi} \frac{(N + \frac{1}{2}) \cos(N + \frac{1}{2})s}{\pi \cos \frac{s}{2}} = \frac{2N + 1}{2\pi} = D_N(2k\pi).$$

Furthermore, $D_N$ is $2\pi$-periodic. We will use the integral of $D_N$. We have from Lemma 2 that

$$\int_0^\pi D_N(s)ds = \frac{1}{\pi} \int_0^\pi \left(\frac{1}{2} + \cos s + \cos(2s) + \cdots + \cos(Ns)\right) ds = \frac{1}{2}$$

(since $\int_0^\pi \cos(j)sdx = 0$ for $j = 1, 2, 3, \cdots$). We also have

$$\int_{-\pi}^0 D_N(s)ds = \frac{1}{2}$$

since the function $D_N$ is even.

So far we proved that

$$S_N f(x) = \int_{-\pi}^\pi f(t)D_N(t - x)dt.$$
By using the substitution \( s = t - x \), and by using the \( 2\pi \)-periodicity of \( f \) and of \( D_N \), we rewrite \( S_N f \) as

\[
S_N f(x) = \int_{-\pi-x}^{\pi-x} f(x+s)D_N(s)ds = \int_{-\pi}^{\pi} f(x+s)D_N(s)ds .
\]

Since

\[
S_N f(x) = \int_{-\pi}^{0} f(x+s)D_N(s)ds + \int_{0}^{\pi} f(x+s)D_N(s)ds ,
\]

then to prove that \( \lim_{N \to \infty} S_N f(x) = f_{av}(x) \), it is enough to prove that

\[
\lim_{N \to \infty} \int_{0}^{\pi} f(x+s)D_N(s)ds = \frac{f(x^+)}{2} \quad \text{and} \quad \lim_{N \to \infty} \int_{-\pi}^{0} f(x+s)D_N(s)ds = \frac{f(x^-)}{2} .
\]

For this, we consider the functions \( h(s) \) and \( k(s) \) defined in \([-\pi, \pi]\) by

\[
h(s) = \begin{cases} 
\frac{f(x+s) - f(x^-)}{s} & \text{for } s < 0 \\
\frac{f(x+s) - f(x^+)}{s} & \text{for } s > 0
\end{cases}
\]

and

\[
k(s) = \begin{cases} 
\frac{s}{2 \sin(s/2)} & \text{for } s \neq 0 \\
1 & \text{for } s = 0
\end{cases}
\]

I leave it as an exercise for you to verify that \( k \) and its derivative are continuous on \([-\pi, \pi]\) (use L’Hopital’s rule at \( s = 0 \)). For the function \( h \), it is piecewise smooth in each closed interval not containing \( s = 0 \). At \( s = 0 \), we have

\[
h(0^-) = \lim_{s \to 0^-} \frac{f(x+s) - f(x^-)}{s} = f'(x^-) \quad \text{and} \quad h(0^+) = \lim_{s \to 0^+} \frac{f(x+s) - f(x^+)}{s} = f'(x^+).\]

Hence \( h \) is piecewise continuous on \([-\pi, \pi]\).

**Claim.** We have the following

\[
\lim_{N \to \infty} \int_{0}^{\pi} h(s)k(s)\sin(N + \frac{1}{2})s ds = 0 \quad \text{and} \quad \lim_{N \to \infty} \int_{-\pi}^{0} h(s)k(s)\sin(N + \frac{1}{2})s ds = 0 .
\]

**Proof of claim.** We prove the first limit. Let \( \epsilon > 0 \). Since, the integrand is piecewise continuous and uniformly bounded, we can find \( \delta > 0 \) such that

\[
\left| \int_{0}^{\delta} h(s)k(s)\sin(N + \frac{1}{2})s ds \right| < \epsilon \quad \forall N \in \mathbb{Z}^+.
\]

The integrand is piecewise smooth on the interval \([\delta, \pi]\). Lemma 1 implies that

\[
\lim_{N \to \infty} \int_{\delta}^{\pi} h(s)k(s)\sin(N + \frac{1}{2})s ds = 0 .
\]

Hence,

\[
\lim_{N \to \infty} \int_{0}^{\delta} h(s)k(s)\sin(N + \frac{1}{2})s ds \leq \epsilon
\]

Since \( \epsilon > 0 \) is arbitrary, then the limit is 0. The second limit of the claim is proved in a similar way and is left as an exercise.
End of the proof of the theorem. By using \( \int_0^\pi D_N(s)ds = \frac{1}{2} \), we get
\[
\int_0^\pi f(x+s)D_N(s)ds - \frac{f(x^+)}{2} = \int_0^\pi (f(x+s) - f(x^+))D_N(s)ds
\]
\[
= \frac{1}{\pi} \int_0^\pi \frac{f(x+s) - f(x^+)}{s} \sin(N + \frac{1}{2}) ds \sin(N + \frac{1}{2}) ds
\]
\[
= \frac{1}{\pi} \int_0^\pi h(s)k(s) \sin(N + \frac{1}{2}) ds
\]
It follows from this and the claim that
\[
\lim_{N \to \infty} \int_0^\pi f(s+u)D_N(s)ds = \frac{f(x^+)}{2}
\]
A similar argument gives
\[
\lim_{N \to \infty} \int_{-\pi}^0 f(s+u)D_N(s)ds = \frac{f(x^-)}{2}
\]
and completes the proof of the theorem.

**Example 1.** The 2\( \pi \)-periodic function \( f \) defined on \([-\pi, \pi]\) by \( f(x) = |x| \) (the triangular wave function) is continuous on \( \mathbb{R} \). It is therefore equal to its Fourier series for all \( x \in \mathbb{R} \). In particular,
\[
|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \cos(2j+1)x \frac{x}{(2j+1)^2}, \quad \forall x \in [-\pi, \pi].
\]
Fourier can be used to evaluate numerical series. For \( x = 0 \), we obtain
\[
0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \cos(2j+1)x \frac{x}{(2j+1)^2}
\]
Hence,
\[
1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}
\]
**Example 2.** The 2\( \pi \)-periodic function \( f \) defined on \([-\pi, \pi]\) by \( f(x) = 1 \) if \( 0 < x < \pi \) and \( f(x) = -1 \) for \(-\pi < x < 0 \) is continuous everywhere except at the points \( k\pi \), with \( k \in \mathbb{Z} \). Thus \( f(x) \) equal its Fourier series for \( x \neq k\pi \). In particular
\[
1 = \frac{4}{\pi} \sum_{j=0}^{\infty} \sin((2j+1)x) \frac{x}{(2j+1)}, \quad \forall x \in (0, \pi).
\]
For \( x = k\pi \), we have \( f_{av}(k\pi) = 0 \) which is the value of the Fourier series when \( x = k\pi \). We can use this series to evaluate the alternating series \( \sum_{j=0}^{\infty} (-1)^j/(2j+1) \).
Indeed, for \( x = \frac{\pi}{2} \), we get
\[
1 = \frac{4}{\pi} \sum_{j=0}^{\infty} \sin((2j+1)x) \frac{\pi}{2} \frac{\pi}{2} = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)}.
\]
Hence,
\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.
\]
Example 3. The $2\pi$-periodic function $f$ defined on $[-\pi, \pi]$ by $f(x) = x$ if $0 \leq x < \pi$ and $f(x) = 0$ for $-\pi < x \leq 0$ is continuous everywhere except at the points $(2k+1)\pi$, with $k \in \mathbb{Z}$. The Fourier series of $f$ is therefore equal to $f(x)$ everywhere except at the points $(2k+1)\pi$. We have then,

$$
\frac{\pi}{4} - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \sin jx}{j} = \begin{cases} 
 0 & \text{if } 0 \leq x < \pi \\
 0 & \text{if } -\pi < x \leq 0
\end{cases}.
$$

At the points $(2k+1)\pi$, we have the average value $f_{av}((2k+1)\pi) = \pi/2$. At such points the Fourier series is $\pi/2$.

3. Differentiation of Fourier Series

When dealing with series of functions, one has to be careful on how to use termwise differentiation. Consider the function of example 2 defined by $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$. We found the Fourier series of $f$. We have in particular

$$
1 = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)} , \quad \forall x \in (0, \pi).
$$

We are tempted to differentiate and write

$$
0 = \frac{4}{\pi} \sum_{j=0}^{\infty} \cos(2j+1)x.
$$

But this cannot be the case since the series diverges. To be able to use term by term differentiation we need an extra condition on $f$. More precisely, we have

**Theorem.** Let $f$ be $2\pi$-periodic and continuous function on $\mathbb{R}$ such that its derivative $f'$ is piecewise smooth. Let $a_0, a_1, b_1, a_2, b_2, \ldots$ be the Fourier coefficients of $f$. Then the Fourier coefficients $a'_0, a'_1, b'_1, a'_2, b'_2, \ldots$ of $f'$ are

$$
a'_0 = 0, \quad a'_n = nb_n, \quad b'_n = -na_n
$$

**Remark.** Under the hypotheses of the theorem, we have term by term differentiation. So if $f$ is continuous, $2\pi$-periodic and $f'$ is piecewise smooth, then

$$
f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx
$$

and

$$
f'(x) \sim \sum_{n=0}^{\infty} nb_n \cos nx - na_n \sin nx
$$

**Claim.** If $g$ is continuous on an interval $[a, b]$ and $g'$ is piecewise continuous on $[a, b]$, then

$$
\int_a^b g'(x)dx = g(b) - g(a).
$$
Proof of the claim. Let \( c_0 = a < c_1 < c_2 < \cdots < c_n = b \) be the possible jump discontinuities of \( g' \). Then,

\[
\int_a^b g'(x)dx = \sum_{j=1}^n \int_{c_{j-1}}^{c_j} g'(x)dx = \sum_{j=1}^n [g(c_j^-) - g(c_{j-1}^+)]
\]

\[
= \sum_{j=1}^n [g(c_j) - g(c_{j-1})]
\]

\[
= (g(c_1) - g(c_0)) + (g(c_2) - g(c_1)) + \cdots + (g(c_n) - g(c_{n-1}))
\]

\[
= g(c_n) - g(c_0) = g(b) - g(a).
\]

Proof of theorem. We use the claim to compute the Fourier coefficients of \( f' \)

\[
a_0' = \frac{1}{\pi} \int_0^{2\pi} f'(x)dx = \frac{f(2\pi) - f(0)}{\pi} = 0
\]

(since \( f \) is \( 2\pi \)-periodic). For \( a_n \ (n \geq 1) \) we use integration by parts

\[
a_n' = \frac{1}{\pi} \int_0^{2\pi} f'(x)\cos nx dx = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos nx dx
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(x)\sin nx dx = nb_n
\]

I leave it as an exercise for you to check that \( b_n' = -na_n \).

Example. Consider the triangular wave function of Example 1 of the previous section. It is defined on \([-\pi, \pi]\) by \( f(x) = |x| \). This function is continuous on \( \mathbb{R} \) and is \( 2\pi \) periodic. Furthermore its derivative \( f'(x) \) is piecewise smooth: it is 1 for \( 0 < x < \pi \) and \(-1 \) for \(-\pi < x < 0 \). \( f' \) is the function of example 2 of the previous section. We have found

\[
|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^2}, \quad \forall x \in [-\pi, \pi].
\]

and by the Theorem we can differentiate term by term to get

\[
f'(x) \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j+1)x)}{(2j+1)}.
\]

4. Integration of Fourier Series

In general an antiderivative of a periodic function is not periodic. For example \( f(x) = 1 \) is periodic (of any period) but its antiderivatives \( F(x) = x + C \) are not periodic. The following lemma gives a necessary and sufficient condition for an antiderivative to be periodic.

Lemma. Let \( f \) be a \( T \)-periodic and piecewise continuous function on \( \mathbb{R} \). The antiderivative \( F \) of \( f \) defined by

\[
F(x) = \int_0^x f(t)dt
\]
is \( T \)-periodic if and only if
\[
\int_0^T f(t) dt = 0.
\]

**Proof.** Suppose that \( f \) satisfies \( \int_0^T f(t) dt = 0 \). We need to show that \( F(x + T) = F(x) \). We have
\[
F(x + T) - F(x) = \int_x^{x+T} f(t) dt - \int_x^x f(t) dt = \int_x^{x+T} f(t) dt = \int_0^T f(t) dt = 0.
\]
Conversely, if the antiderivative is \( T \)-periodic, then
\[
0 = F(T) - F(0) = \int_0^T f(t) dt.
\]

We get the following result about term by term integration of Fourier series.

**Theorem.** Let \( f \) be a piecewise smooth and \( 2\pi \)-period function satisfying
\[
\int_0^{2\pi} f(x) dx = 0 \quad \Leftrightarrow \quad a_0 = 0.
\]
Consider the antiderivative of \( f \) defined by \( F(x) = \int_0^x f(t) dt \). Then the Fourier series of \( F \) is obtained from that of \( f \) by termwise integration. That is, if
\[
f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,
\]
Then,
\[
F(x) = A_0 + \sum_{n=1}^{\infty} \left( -\frac{b_n}{n} \cos nx + \frac{a_n}{n} \sin nx \right),
\]
where \( A_0 = \sum_{n=1}^{\infty} \frac{b_n}{n} \).

**Proof.** The antiderivative \( F \) is continuous and it is also \( 2\pi \)-periodic (see Lemma). Since \( F' = f \) is piecewise smooth, then we can apply the previous Theorem about differentiation of Fourier series. Consider the Fourier series of \( F \):
\[
F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx.
\]
Then
\[
F'(x) \sim \sum_{n=1}^{\infty} nB_n \cos nx - nA_n \sin nx
\]
and so
\[
(nB_n = a_n, \quad -nA_n = b_n) \Leftrightarrow (A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n}).
\]
The coefficient \( A_0 \) can be found by using \( F(0) = 0 \) and equating it to the value of the Fourier series at \( x = 0 \).
Example. We start with the rectangular wave function of example 2 of the previous section: $f$ is defined by $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$ and $f$ is $2\pi$-periodic function. We have found

$$f(x) \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \sin(2j+1)x \quad (2j+1).$$

Since $a_0 = 0$, we can integrate term by term to obtain the Fourier series of $F(x) = \int_0^x f(t)dt$

$$F(x) = A_0 - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2}, \quad \forall x \in [-\pi, \pi].$$

with $A_0 = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2}$. In fact $F(x) = |x|$ for $|x| \leq \pi$ and $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|dx = \frac{\pi}{2}$. We have recovered again

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2}, \quad \forall x \in [-\pi, \pi].$$

For the $2\pi$-periodic function $F(x) = |x|$ on $[-\pi, \pi]$, termwise integration of its Fourier series is not a pure trigonometric series but will contain an extra term, a contribution from $A_0$, since $\int_{-\pi}^{\pi} F(x)dx \neq 0$. More precisely, for $0 < x < \pi$, we have

$$\int_{0}^{x} t dt = \int_{0}^{x} \frac{\pi}{2} dt - \frac{4}{\pi} \sum_{j=0}^{\infty} \int_{0}^{x} \frac{\cos(2j+1)t}{(2j+1)^2} dt ,$$

$$\frac{x^2}{2} - \frac{\pi x}{2} = - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)^3}$$

Thus, for $0 < x < \pi$, we have

$$\frac{x^2}{2} - \frac{\pi x}{2} = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)^3}.$$
In fact this allows us to obtain an expansion of \( x^2 \) over \([0, \pi]\) by using the series for \( x \) and for \((x^2 - \pi x)/2\). We have

\[
x^2 = 2\left(\frac{x^2 - \pi x}{2}\right) + \pi x
\]

\[
= 2\left(-\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j + 1)x)}{(2j + 1)^3}\right) + \pi \left(\frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j + 1)x)}{(2j + 1)^2}\right)
\]

\[
= \frac{\pi^2}{2} - 4 \sum_{j=0}^{\infty} \frac{\cos((2j + 1)x)}{(2j + 1)^2} + \frac{2\sin((2j + 1)x)}{\pi(2j + 1)^3}.
\]

5. Uniform convergence of Fourier series

A sequence of functions \(g_n(x)\) defined on an interval \(I\) is said to converge uniformly to a function \(g(x)\) on \(I\) if the following holds.

\[
\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \ |g_n(x) - g(x)| < \epsilon, \ \forall x \in I, \ \forall n \geq N.
\]

This means that for any given \(\epsilon > 0\), we can find \(N\) that depends only on \(\epsilon\) so that

\[
g_n(x) \text{ is within } \epsilon \text{ from } g(x) \text{ for all } x \in I \text{ and for all } n > N.
\]

A sequence of functions \(h_n(x)\) defined on an interval \(I\) is said to converge pointwise to a function \(h(x)\) on \(I\) if the following holds.

\[
\forall x \in I, \ \forall \epsilon > 0, \ \exists N \in \mathbb{Z}^+, \ |h_n(x) - h(x)| < \epsilon, \ \forall x \in I, \ \forall n \geq N.
\]

In the pointwise convergence \(N\) depends on \(\epsilon\) and on \(x\).

The uniform convergence is stronger than pointwise convergence. A particular consequence of the uniform convergence is the following. If each \(g_n\) is continuous in an interval \(I\) and if \(g_n \rightarrow g\) uniformly in \(I\), then \(g\) is also continuous.

A series of function \(\sum_{n=1}^{\infty} f_n(x)\) converges uniformly to a function \(f\) on an interval \(I\) if the sequence of partial sums \(s_n(x) = \sum_{j=1}^{n} f_j(x)\) converges uniformly to \(f(x)\).

The following two propositions give sufficient conditions for the uniform convergence of Fourier series.

**Proposition 1.** Let

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]
be the Fourier series of a piecewise smooth and $2\pi$-periodic function $f$. If $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$, then the Fourier series converges uniformly.

Since the partial sums of the Fourier series are continuous, the proposition implies that the limit of the Fourier series is a continuous function. Thus the function $f$ is continuous everywhere except possibly at removable discontinuities. This means $f(x^+) = f(x^-)$ everywhere. Note also that if $f$ has a jump discontinuity at $x_0$ (i.e. $f(x_0^+) \neq f(x_0^-)$), then the Fourier series of $f$ does not converge uniformly on any interval containing $x_0$.

**Proposition 2.** If a function $f$ is continuous in $\mathbb{R}$, is piecewise smooth, and $2\pi$-periodic, then its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

converges uniformly to $f$ on $\mathbb{R}$.

6. **Exercises**

In the following exercises, a $2\pi$-periodic function $f$ is given on the interval $[-\pi, \pi]$. (a.) Find the Fourier series of $f$; (b.) Find the intervals where $f(x)$ is equal to its Fourier series; (c.) Determine whether the Fourier series converges uniformly

**Exercise 1.** $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 1 & \text{if } -\pi/2 < x < 0 \\ 0 & \text{if } \pi/2 < |x| < \pi \end{cases}$

**Exercise 2.** $f(x) = |\sin x|$. Use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

**Exercise 3.** $f(x) = |\cos x|$. Use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1}$.

**Exercise 4.** $f(x) = \cos^2 x$ (thing about a trig. identity)

**Exercise 5.** $f(x) = \sin^2 x$

**Exercise 6.** $f(x) = x^2$. Use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

**Exercise 7.** $f(x) = x(\pi - |x|)$. Use the Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$.

**Exercise 8.** Use the Fourier series for $x^2$ that you found in exercise 6 to deduce the fourier series of $x^4 - \pi^2 x^2$ on $[-\pi, \pi]$ (use integration of Fourier series).

**Exercise 9.** Use the Fourier series you found in exercise 8. To deduce that

$$x^4 - 2\pi^2 x^2 = -\frac{7\pi^4}{15} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \cos nx \quad \text{for } -\pi < x < \pi.$$
Deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

**Exercise 10.** Suppose that $f(x)$ has Fourier series $\sum_{n=1}^{\infty} e^{-n^2} \sin nx$. Find the Fourier series of $f'(x)$ and the Fourier series of $f''(x)$ (justify your answer).

**Appendix**

In this appendix, we prove Propositions 1 and 2 about uniform convergence of Fourier series. Given a series $\sum f_n(x)$ of functions, a practical test for uniform convergence is the following.

**Weierstrass M-Test.** Given a series of functions $\sum f_n(x)$ on an interval $I$. If there is a sequence of real numbers $M_n \geq 0$ such that $|f_n(x)| \leq M_n$, $\forall x \in I$, $\forall n \in \mathbb{Z}^+$ and if $\sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $I$.

**Schwartz Inequalities.** Given series of real numbers $\sum \alpha_n$ and $\sum \beta_n$ so that $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n^2 < \infty$, then

$$\left| \sum_{n=1}^{\infty} \alpha_n \beta_n \right| \leq \left( \sum_{n=1}^{\infty} \alpha_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \beta_n^2 \right)^{1/2}.$$

Given piecewise continuous functions $f$ and $g$ on an interval $[a, b]$, then we have

$$\left| \int_a^b f(x)g(x)dx \right| \leq \left( \int_a^b f(x)^2 dx \right)^{1/2} \left( \int_a^b g(x)^2 dx \right)^{1/2}.$$

**Proof.** We prove the first inequality and leave the second as an exercise. The proof is based on the following observation: if $A$, $B$, and $C$ are real constants such that $Ax^2 + 2Bx + C \geq 0$, $\forall x \in \mathbb{R}$, then necessarily $B^2 - AC \leq 0$.

Now, let $N \in \mathbb{Z}^+$ and define $A_N$, $B_N$, and $C_N$ by

$$A_N = \sum_{n=1}^{N} \alpha_n^2, \quad B_N = \sum_{n=1}^{N} \alpha_n \beta_n, \quad \text{and} \quad C_N = \sum_{n=1}^{N} \beta_n^2.$$

For $x \in \mathbb{R}$, we have

$$\sum_{n=1}^{N} (x\alpha_n + \beta_n)^2 = x^2 \sum_{n=1}^{N} \alpha_n^2 + 2x \sum_{n=1}^{N} \alpha_n \beta_n + \sum_{n=1}^{N} \beta_n^2 \geq 0.$$

Thus,

$$A_N x^2 + 2B_N x + C_N \geq 0, \quad \forall x \in \mathbb{R}$$
and the observation implies that \( B_N^2 \leq A_N C_N \). Since by hypothesis, \( \lim_{N \to \infty} A_N \) and \( \lim_{N \to \infty} B_N \) are finite, we get (after letting \( N \to \infty \))
\[
\left( \sum_{n=1}^{\infty} \alpha_n \beta_n \right)^2 \leq \sum_{n=1}^{\infty} \alpha_n^2 \sum_{n=1}^{\infty} \beta_n^2 .
\]
The Schwartz inequality is obtained by taking the square root of the above inequality.

**Bessel’s inequality** Let \( f \) be a \( 2\pi \)-periodic and piecewise continuous function with Fourier series
\[
a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx .
\]
Then
\[
\frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} \leq \frac{1}{2\pi} \int_0^{2\pi} f(x)^2 \, dx .
\]

**Proof.** Let \( S_N f \) be the \( N \)-th partial sum of the Fourier series of \( f \). We have
\[
||f(x) - S_N f(x)||^2 = < f - S_N f, f - S_N f > = ||f||^2 - 2 < f, S_N f > + ||S_N f||^2 .
\]
Now
\[
f(x)S_N f(x) = \frac{a_0}{2} f(x) + \sum_{n=1}^{N} a_n \cos(nx)f(x) + b_n \sin(nx)f(x) .
\]
To find \( < f, S_N f > \), we integrate both sides from 0 to \( 2\pi \) and use the fact that
\[
a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx, \quad \frac{a_n}{b_n} = \frac{1}{\pi} \int_0^{2\pi} f(x) \left\{ \frac{\cos nx}{\sin nx} \right\} dx
\]
to obtain
\[
< f, S_N f > = 2\pi \left( \frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right) .
\]
To find \( ||S_N f||^2 \), we use the orthogonality of the trigonometric system to obtain
\[
\int_0^{2\pi} S_N f(x)^2 \, dx = 2\pi \frac{a_0^2}{4} + \sum_{n=1}^{N} a_n^2 ||\cos nx||^2 + b_n^2 ||\sin nx||^2
\]
\[
= 2\pi \left( \frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right) .
\]
These equalities imply that
\[
0 \leq ||f(x) - S_N f(x)||^2 = ||f||^2 - 2\pi \left( \frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right)
\]
and then
\[
\frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \leq \frac{1}{2\pi} ||f||^2 = \frac{1}{2\pi} \int_0^{2\pi} f(x)^2 \, dx .
\]
The Bessel’s inequality follows by letting \( N \to \infty \).

**Proof of Proposition 1.** Suppose that the Fourier coefficients of \( f \) satisfy \( \sum_n |a_n| + |b_n| < \infty \). We use the Weierstrass M-test to show that the Fourier
series \( \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \) converges uniformly on \( \mathbb{R} \). For this, we just need to take \( M_n = |a_n| + |b_n| \) and observe that
\[ |a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n|. \]

**Proof of Proposition 2.** Suppose that \( f \) is continuous piecewise smooth and \( 2\pi \)-periodic. Let
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{and} \quad \sum_{n=1}^{\infty} a'_n \cos nx + b'_n \sin nx \]
be the Fourier series of \( f \) and of \( f' \). We need to show that the first series converge uniformly. For this it is enough to show that the series \( \sum_{n=1}^{\infty} |a_n| + |b_n| < \infty \) (the Weierstrass M-test again would imply uniform convergence).

We apply Bessel’s inequality to \( f' \):
\[ \sum_{n=1}^{\infty} (a'_n)^2 + (b'_n)^2 < \frac{1}{\pi} \int_0^{2\pi} f'(x)^2 \, dx \]

We know that \( a'_n = nb_n \) and \( b'_n = -na_n \). Hence,
\[ \sum_{n=1}^{\infty} (|a_n| + |b_n|) = \sum_{n=1}^{\infty} \left( \frac{|a_n|}{n} + \frac{|b_n|}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n} (|a'_n| + |b'_n|) \]
\[ \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n=1}^{\infty} (|a'_n| + |b'_n|)^2 \right)^{1/2} \quad \text{(Schwartz inequality)} \]
\[ \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} 2 \left( \sum_{n=1}^{\infty} (|a'_n|^2 + |b'_n|^2) \right)^{1/2} \]
\[ \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} 2 \frac{1}{\pi} \int_0^{2\pi} f'(x)^2 \, dx \]

This proves the uniform convergence of the Fourier series of \( f \).