Ch.4.1 # 4 (a) \( A \times (B \cap C) = \{1,2,3\} \times (\{1,4\} \cap \{3,4\}) = \{1,2,3\} \times \{4\} \)
\[= \{(1,4),(2,4),(3,4)\}\]
\(\{A \times B\} \cap \{A \times C\} = (\{1,2,3\} \times \{1,4\}) \cap (\{1,2,3\} \times \{3,4\})\)
\[= \{(1,1),(1,4),(2,1),(2,4),(3,1),(3,4)\}\]
\[\cap \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}\]
\[= \{(1,4),(2,4),(3,4)\}\]
The rest of the theorem can be verified just as easily.

#5 (a) Let \( \langle a, b \rangle \in A \times (B \cup C) \). Then \( a \in A \) and \( b \in B \cup C \).
So \( a \in A \), and \( b \in B \) or \( b \in C \). Now if \( b \in B \), then
\( \langle a, b \rangle \in A \times B \). And if \( b \in C \), then \( \langle a, b \rangle \in A \times C \).
\( \therefore \langle a, b \rangle \in A \times B \) or \( \langle a, b \rangle \in A \times C \). ...
\( \therefore \langle a, b \rangle \in (A \times B) \cup (A \times C) \)
So \( A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \) .... (1)
Now let \( \langle a, b \rangle \in (A \times B) \cup (A \times C) \). Then either \( \langle a, b \rangle \in A \times B \) or \( \langle a, b \rangle \in A \times C \). In the first case \( a \in A \) and \( b \in B \).
So \( a \in A \) and \( b \in B \cup C \). So \( a \in A \) and \( b \in (B \cup C) \).
And in the second case \( a \in A \) and \( b \in C \). So \( a \in A \) and \( b \in C \).
So \( a \in A \) and \( b \in (B \cup C) \). Thus in either case
\( \langle a, b \rangle \in A \times (B \cup C) \). Hence \( (A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \) .... (2)
From (1) \& (2) it follows that \( A \times (B \cup C) = (A \times B) \cup (A \times C) \).

(b) Let \( \langle a, b \rangle \in (A \times B) \cap (C \times D) \). Then \( \langle a, b \rangle \in A \times B \) and
\( \langle a, b \rangle \in C \times D \). \( \therefore \ a \in A \) and \( b \in B \), and \( a \in C \)
and \( b \in D \). \( \therefore \ a \in A \cap C \) and \( b \in B \cap D \). So
\( \langle a, b \rangle \in (A \cap C) \times (B \cap D) \). \( \therefore \ (A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D) \).
Now let \( \langle a, b \rangle \in (A \cap C) \times (B \cap D) \). Then \( a \in A \cap C \) and
\( b \in B \cap D \). So \( a \in A \) and \( a \in C \), and \( b \in B \) and \( b \in D \).
So \( \langle a, b \rangle \in A \times B \) and \( \langle a, b \rangle \in C \times D \). \( \therefore \ (A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D) \). Hence \( (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \).
Ch. 4.1 #6. The first 3 sentences are okay. The third sentence is logically equivalent to
\[(x \in A \lor x \in C) \land (y \in B \lor y \in D)\]
which is logically equivalent to
\[(x \in A \land y \in B) \lor (x \in A \land y \in D) \lor (x \in C \land y \in B) \lor (x \in C \land y \in D)\].
So we have 4 cases to consider. The "proof" only considers 2 of these 4 cases.

#7. \(|A \times B| = m \cdot n\)

#10. We want to prove that \((A \times B) \cap (C \times D) = \emptyset\) implies \(A \cap C = \emptyset\) or \(B \cap D = \emptyset\). It will suffice to prove the contrapositive. Now \(-((A \cap C = \emptyset) \lor (B \cap D = \emptyset))\) is equivalent to \(A \cap C \neq \emptyset\) and \(B \cap D \neq \emptyset\).
So suppose \(A \cap C \neq \emptyset\) and \(B \cap D \neq \emptyset\). Then we can
an element \(a \in A \cap C\) and an element \(b \in B \cap D\).
So \((a,b) \in A \times B\) because \(a \in A\) and \(b \in B\)
and \((a,b) \in C \times D\) because \(a \in C\) and \(b \in D\),
Hence \((a,b) \in (A \times B) \cap (C \times D)\) and so \((A \times B) \cap (C \times D) \neq \emptyset\).
So \(-((A \cap C = \emptyset) \lor (B \cap D = \emptyset))\) implies \((A \times B) \cap (C \times D) \neq \emptyset\).
\[
\therefore (A \times B) \cap (C \times D) = \emptyset \text{ implies } A \cap C = \emptyset \text{ or } B \cap D = \emptyset.
\]

Ch. 4.2 #1. (a) domain = set of all parents with a living offspring
range = set of all offsprings with a living parent
2 (a) domain = set of all males with a living sibling
range = set of people with a living brother.
1 (b) domain = \(\mathbb{R}\), range = \(\mathbb{R}^+ = (0, \infty)\)
2 (a) domain = \((-\infty, 1] \cup [1, \infty)\), range = \((-1, 1)\)
4.2 #4

\[ R = \{\langle 1,4 \rangle, \langle 4,5 \rangle, \langle 2,5 \rangle, \langle 3,6 \rangle\} \]

\[ S = \{\langle 4,5 \rangle, \langle 4,6 \rangle, \langle 5,4 \rangle, \langle 6,6 \rangle\} \]

(a) \[ S \circ R = \{\langle 1,5 \rangle, \langle 1,6 \rangle, \langle 1,4 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle\} \]

(b) \[ S \circ S = \{\langle 4,4 \rangle, \langle 5,5 \rangle, \langle 4,6 \rangle\} \]

(c) \[ S^{-1} \circ R = \{\langle 1,5 \rangle, \langle 1,4 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \langle 3,4 \rangle\} \]

(d) \[ R^{-1} \circ S = \{\langle 4,1 \rangle, \langle 4,2 \rangle, \langle 4,3 \rangle, \langle 5,1 \rangle, \langle 6,3 \rangle\} \]

\[ = (S^{-1} \circ R)^{-1} \]

#7 \[ p \text{ Eq } q \text{ Er } \rightarrow p \text{ Fr} \]

#9 (a) TRUE. Let \( \langle a,b \rangle \in R \). Then \( a \in \text{dom}(R) \) and \( b \in \text{ran}(R) \). So \( \langle a,b \rangle \in \text{dom}(R) \times \text{ran}(R) \).

Since this is true for any \( \langle a,b \rangle \in R \), it follows that \( R \subseteq \text{dom}(R) \times \text{ran}(R) \).

(b) TRUE. Let \( \langle b,a \rangle \in R^{-1} \). Then \( \langle a,b \rangle \in R \).

So \( \langle a,b \rangle \in S \) because \( R \subseteq S \). Hence \( \langle b,a \rangle \in S^{-1} \). Hence \( R^{-1} \subseteq S^{-1} \).

(c) Let \( \langle b,a \rangle \in (R \cup S)^{-1} \). Then \( \langle a,b \rangle \in R \cup S \).

So \( \langle a,b \rangle \in R \) or \( \langle a,b \rangle \in S \). In the first case \( \langle b,a \rangle \in R^{-1} \) and in the second case \( \langle b,a \rangle \in S^{-1} \). So \( \langle a,b \rangle \in R^{-1} \cup S^{-1} \). Hence \( (R \cup S)^{-1} = R^{-1} \cup S^{-1} \).

4.3 #1

Diagram:

- Nodes: a, b, c, d, e
- Edges:
  - a to b, a to c, a to d, a to e
  - b to c, b to d, b to e
  - c to d, c to e
  - d to e

Words:
- bad
- bed
- cab
(b) $R$ is reflexive

(c) $R$ is symmetric

(d) $R$ is not transitive

\[ \langle \text{rat}, \text{bird} \rangle \in R \land \langle \text{bird}, \text{dog} \rangle \in R \text{ but } \langle \text{rat}, \text{dog} \rangle \notin R. \]

3. \[ i_A = \begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
4
\end{array} \]

4. (a) $R_1 = \{ \langle a, c \rangle, \langle b, d \rangle, \langle c, c \rangle, \langle d, a \rangle, \langle d, b \rangle \}$

not reflexive, not symmetric, not transitive

$a,a \notin R_1; \ (a,c) \in R_1 \text{ but } (c,a) \notin R_1; \ (d,a) \in R_1 \land (a,c) \in R_1 \text{ but } (d,c) \notin R_1$

(b) $R_2 = \{ \langle a, b \rangle, \langle a, d \rangle, \langle b, a \rangle, \langle b, d \rangle \}$

not reflexive, not symmetric, not transitive

$a,a \in R_2; \ (b,d) \in R_2 \text{ but } (d, b) \notin R_2; \ (a, b) \in R_2 \land (b, a) \in R_2 \text{ but } (a, b) \notin R_2$

(c) $R_3 = \{ \langle a, a \rangle, \langle b, b \rangle, \langle b, d \rangle, \langle c, c \rangle, \langle d, b \rangle, \langle d, d \rangle \}$

reflexive, symmetric, transitive

(d) $R_4 = \{ \langle a, b \rangle, \langle a, c \rangle, \langle a, d \rangle, \langle b, c \rangle, \langle b, d \rangle \}$

not reflexive, not symmetric, transitive.

$a,a \notin R_4; \ (a,c) \in R_4 \text{ but } (c,a) \notin R_4.$
14(a) Suppose $R$ and $S$ are reflexive on $A$. Then $i_A \subseteq R$ and $i_A \subseteq S$. So $i_A \subseteq R \cap S$

Thus $R \cap S$ will be reflexive.

13(a) Also $i_A \subseteq R \cup S$, so $R \cup S$ will be reflexive.

16(a) Now $i_A = i_A \circ i_A \subseteq R \circ S$, so $R \circ S$ will

also be reflexive. Finally $i_A \subseteq R^{-1}$, so $R^{-1}$ is reflexive.

14(b) Suppose $R$ and $S$ are symmetric. Then

$aRb \Rightarrow bRa \in R$ and $(a,b) \in S \Rightarrow (b,a) \in S$.

Let $(a,b) \in R \cap S$ then $(a,b) \in R$ and $(a,b) \in S$. So

$(b,a) \in R$ and $(b,a) \in S$. So $(b,a) \in R \cap S$. Hence

$R \cap S$ is symmetric.

13(b) Similarly if $(a,b) \in R \cup S$, then $(a,b) \in R$ or $(a,b) \in S$.

In either case we get $(b,a) \in (R \cup S)$. So $R \cup S$ is

also symmetric.

Now suppose $(a,b) \in R \circ S$. Then $\exists c \in A$ such

that $(a,c) \in S$ and $(c,b) \in R$. So $(a,b) \in S$ and

$(b,c) \in R$, so $(b,a) \in S \circ R$. So it seems that

we can't prove that $(b,a) \in R \circ S$. So let us

look for a counterexample.

16(b) Let $S = \{<1,1>\}$ and $R = \{<1,2>, <2,1>\}$. Then

$R \circ S = \{<1,2>\}$. $R$ and $S$ are symmetric

but $R \circ S$ is not symmetric.

12(b) Finally $R$ is symmetric iff $R = R^{-1}$. Since

$R$ is symmetric, $R = R^{-1}$. So $R^{-1} = (R^{-1})^{-1}$. So $R^{-1}$ is symmetric.
Suppose \( R \) and \( S \) are transitive. Then
\[
(a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R \quad \land \quad (a, b) \in S \land (b, c) \in S \rightarrow (a, c) \in S.
\]

14(c) Suppose \( (a, b) \in R \land S \land (b, c) \in R \land S \). Then \( (a, b) \in R \land (b, c) \in R \). So \( (a, c) \in R \). Also \( (a, b) \in S \land (b, c) \in S \). So \( (a, c) \in S \). So \( (a, c) \in R \land S \). So \( R \land S \) will be transitive.

13(c) Let \( R = \{1, 2\} \) and \( S = \{2, 3\} \). Then \( R \) and \( S \) are transitive. But \( R \land S = \{1, 2, 3\} \) is not transitive because \( 1, 2 \in R \land S \) and \( 2, 3 \in R \land S \) but \( 1, 3 \notin R \land S \).

12(c) Suppose \( (a, b) \in R^{-1} \) and \( (b, a) \in R^{-1} \). Then \( (a, b) \in R \land (b, c) \in R \). So \( (a, c) \in R \) because \( R \) is transitive. Hence \( (c, a) \in R^{-1} \). So \( (c, b) \in R \) and \( (b, a) \in R^{-1} \) implies \( (c, a) \in R^{-1} \). i.e. \( R^{-1} \) is transitive.

16(c) Finally let \( S = \{1, 2\} \), \( 3, 4\} \) and \( R = \{2, 3\} \), \( 4, 5\} \). We can then check that \( S \) and \( R \) are transitive. Also \( R \land S = \{1, 3\} \), \( 3, 5\} \). So \( R \land S \) is not transitive because \( 1, 3 \in R \land S \) and \( 3, 5 \in R \land S \) but \( 1, 5 \notin R \land S \). Hence if \( R \) and \( S \) are transitive, it does not follow that \( R \land S \) is transitive.

Qu16. Suppose \( R \) and \( S \) are relations. Prove or give a counter-example
(a) If \( R \) and \( S \) are reflexive, must \( R \land S \) also be reflexive
(b) If \( R \) and \( S \) are symmetric, must \( R \land S \) also be symmetric
(c) If \( R \) and \( S \) are transitive, must \( R \land S \) also be transitive.
1. \( P_1 = \{ \{1\}, \{2\}, \{3\} \}, \quad P_2 = \{ \{1\}, \{2, 3\} \}, \quad P_3 = \{ \{2\}, \{1, 3\} \}, \quad P_4 = \{ \{3\}, \{1, 2\} \}, \quad P_5 = \{ \{1, 2, 3\} \}. \) Note each \( P_i \) is related to the corresponding equiv. relation \( R_i \) below, \((i=1,\ldots,5)\).

2. \( R_1 = \{ \langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle \}, \quad R_2 = \{ \langle 1,1 \rangle, \langle 2,2 \rangle, \langle 2,3 \rangle, \langle 3,2 \rangle, \langle 3,3 \rangle \} \)
   \( R_3 = \{ \langle 2,2 \rangle, \langle 1,1 \rangle, \langle 1,3 \rangle, \langle 3,1 \rangle, \langle 3,3 \rangle \} \)
   \( R_4 = \{ \langle 3,3 \rangle, \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle \} \)
   \( R_5 = \{ \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle, \langle 1,3 \rangle, \langle 3,1 \rangle, \langle 2,3 \rangle, \langle 3,2 \rangle, \langle 3,3 \rangle \} \)

3. (a) \( R \) is an equivalence relation.
   \( C_a = \) set of all words beginning with "a"
   \( C_b = \) set of all words beginning with "b"
   \( C_z = \) set of all words beginning with "z"

   The equivalence classes are \( C_a, C_b, \ldots, C_z \).

3. (c) \( T \) is an equiv. relation; \( C_n = \) words with \( n \) letters.

3. (b) \( S \) is not an equivalence relation because \( S \) is not transitive. \( \langle go, to \rangle \in S \) and \( \langle to, at \rangle \in S \) but \( \langle go, at \rangle \notin S \).

4. (a) \( R \) is not an equiv. relation because it is not symmetric.

4. (b) \( S = \{ \langle x, y \rangle \in R \times R : x - y \in \mathbb{Q} \} \) is an equivalence relation. Verify that \( R \) is reflexive, symmetric and transitive. The equivalence classes are \( C_x = \{ x + q : q \in \mathbb{Q} \} \). For each \( x \in R \), \( C_x \) will be an equivalence class - but some of these will coincide. For example:

   \[ C_{\sqrt{2} + 1} = C_{\sqrt{2} - 3} = C_{\sqrt{2}}. \]
   \[ C_0 = C_1 = C_{\sqrt{2} + 2} = C_{3/4} = C_{2/3}. \]
Ch. 4.6 #4 (c)  \( T = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : (\exists n \in \mathbb{Z}) (y = x \cdot 10^n) \} \)

For each \( x \in \mathbb{R} \), \( x = x \cdot 10^0 \). So \( (x, x) \in T \) for each \( x \in \mathbb{R} \). So \( T \) is reflexive.

Suppose \( (x, y) \in T \). Then \( (\exists n \in \mathbb{Z}) (y = x \cdot 10^n) \).
\( So \ x = y \cdot 10^{-n} \). Hence \( (y, x) \in T \). So \( T \) is symmetric.

Suppose \( (x, y) \in T \) and \( (y, z) \in T \). Then \( (\exists n \in \mathbb{Z}) (y = x \cdot 10^n) \) and \( (\exists k \in \mathbb{Z}) (z = y \cdot 10^k) \). So \( z = y \cdot 10^k = (x \cdot 10^n) \cdot 10^k = x \cdot 10^{n+k} \). Hence \( (x, z) \in T \). So \( T \) is transitive.

\( \therefore \) \( T \) is an equivalence relation.

Let \( C_x = \{ x \cdot 10^n : n \in \mathbb{Z} \} \). Then for each \( x \in \mathbb{R} \), \( C_x \) will be an equivalence class. Some of these classes will coincide. For example
\[ C_1 = C_{10} = C_{100} = C_{1/10} = C_{1/100} = C_{1/1000} \]
\[ C_{\sqrt{2}} = C_{10 \sqrt{2}} = C_{100 \sqrt{2}} = C_{\sqrt{2}/10} = C_{\sqrt{2}/100} \].

Suppose \( R \) and \( S \) are equivalence relations. Then \( R \) and \( S \) are both reflexive, symmetric and transitive.

19 (a) From Problems 4.3 #12-16, it follows that \( R S \) will be reflexive, symmetric & transitive. So \( R S \) will be an equivalence relation.

27 (a) \( R S \) will not always be an equiv. relation because \( R S \) will not always be transitive.

28 (a) \( R S \) will not always be an equiv. relation because \( R S \) will not always be symm. or trans.
Ch. 4.6 #10. (a) Let \( x \in \mathbb{Z} \). Then \( x - x = 0 = 0 \cdot m \). So 
\[ \langle x, x \rangle \in C_m. \] So \( C_m \) is reflexive.

Suppose \( \langle x, y \rangle \in C_m \). Then \((\exists k \in \mathbb{Z})\) such that \( x - y = km \). So 
\[ y - x = (-k)m \]
Hence \( \langle y, x \rangle \in C_m \). So \( C_m \) is symmetric.

It was already shown on p. 221 that \( \langle x, y \rangle \in C_m \) and \( \langle y, z \rangle \in C_m \) implies \( \langle x, z \rangle \in C_m \).
So \( C_m \) is also transitive. Hence \( C_m \) is an equivalence relation.

(b) \( C_2 \): \[ [0]_2 = \{2k: k \in \mathbb{Z}\} \] and \[ [1]_2 = \{2k+1: k \in \mathbb{Z}\} \] are the equiv. classes

\( C_3 \): \[ [0]_3 = \{3k: k \in \mathbb{Z}\}, \]
\[ [1]_3 = \{3k+1: k \in \mathbb{Z}\}, \]
and \[ [2]_3 = \{3k+2: k \in \mathbb{Z}\} \] are the equiv. classes.

\( C_m \) will have \( m \) equiv. classes: \([0]_m, \ldots, [m-1]_m\).

24(a) Suppose that \( R \) is a transitive, reflexive relation on \( A \). Then \( i_A \in R \), \( i_A \in R \). Also \( \langle x, y \rangle \in R \) and \( \langle y, z \rangle \in R \) implies \( \langle x, z \rangle \in R \). So \( i_A \in R \cap R^{-1} \). Hence \( R \cap R^{-1} \) is reflexive.

Also if \( \langle z, y \rangle \in R^{-1} \) and \( \langle y, x \rangle \in R^{-1} \), Then \( \langle x, y \rangle \in R \) and \( \langle y, x \rangle \in R \). So \( \langle x, z \rangle \in R \) because \( R \) is transitive.
Thus \( \langle z, x \rangle \in R^{-1} \). Hence \( R^{-1} \) is transitive.

From Problem 4.3 #9(c), it follows that \( R \cap R^{-1} \) is also transitive. Finally suppose \( \langle x, y \rangle \in R \cap R^{-1} \). Then \( \langle x, y \rangle \in R \) and \( \langle y, x \rangle \in R^{-1} \). So \( \langle x, y \rangle \in R^{-1} \) and \( \langle y, x \rangle \in R^{-1} \). Hence \( \langle x, x \rangle \in R \cap R^{-1} \). So \( R \cap R^{-1} \) is symmetric. \( \therefore R \cap R^{-1} \) is an equivalence relation.
Ch. 5.1 #1. (a) YES. (b) NO. \( (1, 2) \in f & (1, 3) \in f \) but \( z \neq 3 \). (c) YES.

2 (a) NO, \(-\exists w \in B \langle d, w \rangle \in f\)

(b) \( f \) is not a function \( \langle g_0, g \rangle \) \& \( \langle g_0, 0 \rangle \in f \) but \( g \not\equiv 0 \).

(c) YES, \( R \) is a function from \( P \) to \( P \).

#3 (a) \( f(a) = b \), \( f(b) = b \), \( f(c) = a \)

(b) \( H(\text{Italy}) = \text{Rome} \)

(c) \( f(2) = 2^2 - 2(2) = 0 \)

(d) \( F(\{1, 3\}) = \{1, 2, 3\} - \{1, 3\} = \{2\} \)

\( L_0 H = \text{identity func.} \)

#5 (a) \( L_0 H : N \rightarrow N \) \( (L_0 H)(a \text{ country}) = \text{the same country} \)

(b) \( H_0 L : C \rightarrow C \) \( (H_0 L)(a \text{ city}) = \text{capital of the country in which that city is located} \)

#6 \( f \circ g)(x) = f(g(x)) \)

\[ = f(2x-1) = \frac{1}{(2x-1)^2 + 2} = \frac{1}{4x^2 - 4x + 3} \]

\( (g \circ f)(x) = g(f(x)) \)

\[ = g\left(\frac{1}{x^2 + 2}\right) = \frac{2}{\left(\frac{1}{x^2 + 2}\right)} - 1 = \frac{-x^2}{x^2 + 2} \]

#8 (a) \( i_A \) is clearly an equivalence relation because it is reflexive, symmetric and transitive. \( i_A \) is also a function because \( (\forall a \in A) (\exists! b \in A)(a, b) \in i_A \).

The unique \( b \) is just \( a \).

(b) Suppose \( R \) is an equivalence relation on \( A \) which is a function. Then \( i_A \subseteq R \) because \( R \) is reflexive.

Now if \( R \neq i_A \), then \( \exists \{a \in A \in A \in A \}(a \neq b \land (a, b) \in R \).

So \( (a, a) \in R \) and \( (a, b) \in R \). So \( R \) won't be a function.

Hence \( i_A \) is the only equiv. relation on \( A \) which is a function.
(a) Suppose \( f(x) = f(y) \). Then \( \frac{x+1}{x-1} = \frac{y+1}{y-1} \).

So \( \frac{x-1 + 2}{x-1} = \frac{y-1 + 2}{y-1} \).

\[ \therefore 1 + \frac{2}{x-1} = 1 + \frac{2}{y-1} \]

\[ \therefore \frac{2}{x-1} = \frac{2}{y-1} \]

\[ \therefore x-1 = y-1 \]

\[ \therefore x = y \]

Hence if \( f(x) = f(y) \) then \( x = y \). So \( f \) is one-to-one.

Let \( b \in \mathbb{R} - \{1\} \). Take \( x = \frac{(b+1)}{(b-1)} \). Then

\[ f(x) = \frac{b+1}{b-1} + 1 = \frac{(b+1) + (b-1)}{b-1} = \frac{2b}{b-1} = \frac{2b}{2} = b \]

(b) \( (f \circ f)(x) = f(f(x)) = f\left(\frac{x+1}{x-1}\right) \)

\[ = \frac{x+1}{x-1} + 1 = \frac{(x+1)(x-1)}{x-1} = \frac{x-1}{x+1} - (x-1) \]

\[ = \frac{2x}{2} = x \]

\[ \therefore (f \circ f)(x) = x \]

\[ \therefore f \circ f = i_{\mathbb{R}} \]

6 (a) \( f(2) = \{y \in \mathbb{R} : y^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \)

(b) \( f \) is not one-to-one because \( f(0) = f(-1) \).

\[ f(0) = \{y \in \mathbb{R} : y^2 < 0\} = \emptyset \]

\[ f(-1) = \{y \in \mathbb{R} : y^2 < -1\} = \emptyset \]

(c) \( f \) is not onto because there is no \( x \in \mathbb{R} \) such that \( f(x) = [1, 2] \).

7 (a) \( f([1, 2], [3, 4]) = [1, 2] \cup [3, 4] = [1, 2, 3, 4] \)

(b) not one-to-one. \( f([1, 2]) = f([1], [2]) = [1, 2] \)

(c) onto. Let \( y \in B \). Take \( x = 3y \in A \). Then \( f(x) = y \).
Ch 5.2 #9

(a) \( f: A \to B \) and \( g: B \to C \). Suppose \( f \) is onto and \( g \) is not one-to-one. Then we can find \( b_1, b_2 \in B \) such that \( g(b_1) = g(b_2) \) and \( b_1 \neq b_2 \). Since \( f \) is onto, we can find \( a_1, a_2 \in A \) such that \( f(a_1) = b_1 \) and \( f(a_2) = b_2 \). Also because \( f \) is a function and \( b_1 \neq b_2 \), \( a_1 \neq a_2 \). So 
\[(g \circ f)(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = (g \circ f)(a_2).\]
Since \( a_1 \neq a_2 \), it follows that \( g \circ f \) is not one-to-one.

(b) Suppose \( f \) is not onto and \( g \) is one-to-one. Then we can find an element \( b \in B \) such that there is no \( a \in A \) with \( f(a) = b \). Let \( c = g(b) \). Then there is no element \( b' \in B - \{b\} \) such that \( g(b') = c \). This is because \( g \) is one-to-one. From this it follows that there is no element \( a \in A \) such that \( (g \circ f)(a) = c \) because the only way for us to have \( g(f(a)) = c \) is for \( f(a) \) to be \( b \). But there is no such \( a \). Hence \( g \circ f \) is not onto.

10(a) If \( f \) is one-to-one, then
\[(\forall a_1 \in A)(\forall a_2 \in A) [a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)]\]
Since \( C \subseteq A \), this implies that
\[(\forall a_1 \in C)(\forall a_2 \in C) [a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)]\]
\[\therefore \ f \cap C \text{ will also be one-to-one.}\]

(b) If \( f \cap C \) is onto, then \((\forall b \in B)(\exists a \in C) [f(a) = b] \)
Since \( C \subseteq A \), it follows that \((\forall b \in B)(\exists a \in A) [f(a) = b] \).
\[\therefore \ f \text{ will also be onto.}\]

(c) Let \( A = \mathbb{R}, B = \mathbb{R}, C = [0, \infty) \). Put \( f_1(x) = x^2 \). \( f \) is not one-to-one.
Let \( A = \mathbb{R}, B = \mathbb{R}, C = [0, \infty) \). Put \( f_2(x) = x \). \( f \cap C \) is not onto.
Ch. 5.3 # 3 (a) Suppose \( f(x_1) = f(x_2) \). Then
\[
\frac{2x_1+5}{3} = \frac{2x_2+5}{3}
\]
So \( 2x_1+5 = 2x_2+5 \) \( \therefore 2x_1 = 2x_2 \)
\( \therefore x_1 = x_2 \). Hence \( f \) is one-to-one

(b) Let \( z \in \mathbb{R} \). Put \( x = \frac{3z-5}{2} \). Then
\[
f(x) = \frac{2x+5}{3} = \frac{2 \cdot \frac{3z-5}{2} + 5}{3}
\]
\[
= \frac{3z-5+5}{3} = \frac{3z}{3} = z
\]
\( \therefore f \) is onto.

(c) Let \( y = f^{-1}(x) \). Then \( f(y) = x \). So
\[
\frac{2y+5}{3} = x \quad \therefore 2y+5 = 3x
\]
\( \therefore 2y = 3x-5 \). So \( y = \frac{3x-5}{2} \)
Hence \( f^{-1}(x) = \frac{3x-5}{2} \).

# 6 (a) Suppose \( f(x_1) = f(x_2) \). Then
\[
\frac{3x_1}{x_1-2} = \frac{3x_2}{x_2-2}
\]
\( \therefore \)
\[
\frac{3(x_1-2)+6}{x_1-2} = \frac{3(x_2-2)+6}{x_2-2}
\]
\( \therefore \)
\[
\frac{6}{x_1-2} = \frac{6}{x_2-2}
\]
\( \therefore \)
\[
x_1-2 = x_2-2 \quad \therefore x_1 = x_2
\]
Hence \( f \) is one-to-one

(b) Take \( B = \{R-\{3\}\} \) and let \( z \in B \). Put \( x = \frac{6+z}{3-3} \)
Then \( f(x) = \frac{3 \cdot \left(\frac{6}{3-3}+2\right)}{\left(\frac{6}{3-3}+2\right)-2} \)
\( = \frac{3 \cdot \left(\frac{6}{3-3}+2\right)}{\left(\frac{3}{3-3}+2\right)-2} \)
\( = \frac{3 \cdot \frac{23}{6} - 6 \frac{3}{6}}{6} \)
\( = \frac{3 \cdot 23}{6} = \frac{6 \cdot 23}{6} = 3 \)
\( \therefore f: A \rightarrow B \) is onto.

(c) \( f^{-1}(x) = \left[\frac{6}{x-3}\right]+2 = \frac{2x}{x-3} \).
\( f(x) = \frac{x+7}{5} \), \( f_1(x) = x+7 \), \( f_2(x) = \frac{x}{5} \)

(a) \( (f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(x+7) = \frac{x+7}{5} = f(x) \)

:\( f_2 \circ f_1 = f \)

(b) Let \( y = f^{-1}(x) \). Then \( f(y) = x \). So \( \frac{y+7}{5} = x \)

:\( y+7 = 5x \). So \( y = 5x - 7 \)

:\( f^{-1}(x) = 5x - 7 \).

(c) By inspection, \( f_1^{-1}(x) = x - 7 \) and \( f_2^{-1}(x) = 5x \)

So \((f_1^{-1} \circ f_2^{-1})(x) = f_1^{-1}(f_2^{-1}(x)) = f_1^{-1}(5x) = 5x - 7 \)

:\( (f_2 \circ f_1)^{-1}(x) = (f_1^{-1} \circ f_2^{-1})(x) = 5x - 7 \)

So we have verified that \( f^{-1} = (f_2 \circ f_1)^{-1} \).

11(a) Suppose \( f \) is one-to-one and \( f \circ g = i_B \). Let \( b \) be an arbitrary element of \( B \). Put \( a = g(b) \).

Then \( f(a) = f(g(b)) = (f \circ g)(b) = i_B(b) = b \).

Hence \( f \) is onto. So \( f^{-1} : B \to A \) exists.

Since \( f \circ g = i_B \), it follows that

\( f^{-1} \circ (f \circ g) = f^{-1} \circ i_B \)

:\( (f^{-1} \circ f) \circ g = f^{-1} \)

:\( i_A \circ g = f^{-1} \). So \( g = f^{-1} \).

(b) Suppose \( f \) is onto and \( g \circ f = i_A \). Suppose \( f(x_1) = f(x_2) \). Then \( g(f(x_1)) = g(f(x_2)) \).

So \( (g \circ f)(x_1) = (g \circ f)(x_2) \)

:\( i_A(x_1) = i_A(x_2) \). So \( x_1 = x_2 \)

Hence \( f \) is one-to-one. So \( f^{-1} : B \to A \) exists. Since \( g \circ f = i_A \), it follows that

\( (g \circ f)^{-1} = i_A \circ f^{-1} \)

:\( g \circ (f \circ f^{-1}) = f^{-1} \). So \( g \circ i_B = f^{-1} \). So \( g = f^{-1} \).
Ch. 5.3 #1 (c) Suppose \( f \circ g = i_B \) and \( g \circ f \neq i_A \).
Let \( b \in B \). Put \( a = g(b) \). Then
\[
f(a) = f(g(b)) = (f \circ g)(b) = i_B(b) = b.
\]
So \( f \) is onto. Also if \( f \) was one-to-one, then by part (a), it will follow that \( g = f^{-1} \), and so we would get \( g \circ f = f^{-1} \circ f = i_A \) — a contradiction. Hence \( f \) cannot be one-to-one.

Now suppose \( g(b_1) = g(b_2) \). Then
\[
f(g(b_1)) = f(g(b_2)). \quad \text{So} \quad (f \circ g)(b_1) = (f \circ g)(b_2) \quad \quad \Rightarrow \quad \quad i_B(b_1) = i_B(b_2).
\]
Hence \( b_1 = b_2 \). So \( g \) is one-to-one. Also if \( g \) was onto, then by part (b), it will follow that \( f = g^{-1} \) and from this we would get \( g \circ f = g \circ g^{-1} = i_A \) — a contradiction. Hence \( g \) cannot be onto.

Ch. 5.4 #1 (a) YES. Let be \( f[WUX] \). Then \( \exists a \in WUX \) such that \( f(a) = b \). Since \( a \in WUX \), either \( a \in W \) or \( a \in X \). In the first case be \( f[W] \) and in the second case be \( f[X] \). So be \( f[W] \cup f[X] \). --- (1)

Now suppose be \( f[W] \cup f[X] \). Then be \( f[W] \) or be \( f[X] \). In the first case \( \exists a \in W \) such that \( b = f(a) \). And in the second case \( \exists a \in X \) such that \( b = f(a) \). Hence \( b \in f[WUX] \). --- (2)

From (1) \& (2) it follows that \( f[WUX] = f[W] \cup f[X] \).

(b) NO. Let \( f : \{1, 2\} \to \{3, 2\} \), \( W = \{1\} \) and \( X = \{2\} \).

Then \( f[W-X] = f[\{1\} - \{2\}] = \{3\} \neq \emptyset = \{3\} \cup \emptyset = f[W] \cup f[X] \).
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3. (c) NO. Let \( f : \{1,2\} \rightarrow \{4\} \), \( W = \{1\} \) and \( X = \{2\} \). Then \( f[W] = \{4\} \) and \( f[X] = \{4\} \). So \( f[W] \subseteq f[X] \) but \( W \neq X \). So \( f[W] \not\subseteq f[X] \rightarrow W \not\subseteq X \).

#2

(a) YES. Let \( a \in f^{-1}[Y \cap Z] \). Then \( f(a) \in Y \cap Z \). So \( f(a) \in Y \) and \( f(a) \in Z \). \( \therefore a \in f^{-1}[Y] \) and \( a \in f^{-1}[Z] \). So \( a \in f^{-1}[Y] \cap f^{-1}[Z] \). \( \therefore f^{-1}[Y \cap Z] \subseteq f^{-1}[Y] \cap f^{-1}[Z] \). Now let \( a \in f^{-1}[Y] \cap f^{-1}[Z] \). Then \( a \in f^{-1}[Y] \) and \( a \in f^{-1}[Z] \). So \( f(a) \in Y \) and \( f(a) \in Z \). \( \therefore f(a) \in Y \cap Z \). So \( a \in f^{-1}[Y \cap Z] \). Hence \( f^{-1}[Y] \cap f^{-1}[Z] \subseteq f^{-1}[Y \cap Z] \). \( \therefore f^{-1}[Y \cap Z] = f^{-1}[Y] \cap f^{-1}[Z] \).

(b) YES. The proof is similar to the one in 2(a).

(c) YES. The proof is similar to the one in 2(a).

(d) NO. Let \( f : \{1\} \rightarrow \{2,3,4\} \), \( f(1) = 2 \), \( Y = \{2,3\} \) and \( Z = \{2,4\} \). Then \( f^{-1}[Y] = \{1\} \subseteq \{1\} = f^{-1}[Z] \) but \( Y \cap \{2,3\} \neq \{2,4\} = Z \). So \( f^{-1}[Y] \subseteq f^{-1}[Z] \not\rightarrow Y \subseteq Z \).

#3

NO. Let \( f : \{1,2\} \rightarrow \{3,5\} \), \( f(1) = f(2) = 3 \). Then take \( X = \{1\} \). \( f^{-1}[f[X]] \) \( = f^{-1}[3] = \{1,2\} \neq X \). Now suppose \( a \in X \). Then \( f(a) \in f[X] \). So \( a \in f^{-1}[f[X]] \). \( \therefore X \subseteq f^{-1}[f[X]] \). So \( f^{-1}[f[X]] \supseteq X \) but \( X \neq f^{-1}[f[X]] \) in general.

#4

NO. Let \( f : \{1\} \rightarrow \{2,3\} \), \( f(1) = 2 \), and \( Y = \{2,3\} \). Then \( f[f^{-1}[Y]] = f[\{1\}] = \{2\} \neq Y \). So \( f[f^{-1}[Y]] \neq Y \), in general.