2.12 Prove that if $G$ is a graph of order $n$ such that
\[ \Delta(G) + \delta(G) \geq n - 1, \]
then $G$ is connected and $\text{diam}(G) \leq 2$. Show that bound $n - 1$ is sharp.

Solution:
We shall first prove that $G$ is connected. One of the things that will come out of this will be that $\text{diam}(G) \leq 4$. The next thing we shall do is show that the bound $n - 1$ is sharp, and next we shall provide a simple example of a connected graph satisfying the hypotheses with $\text{diam}(G) = 3$. This example is a result of solving Problem 2.10(a) and reveals that $\text{diam}(G) \leq 2$ is not a consequence of the hypotheses. Finally, we shall provide an example $G$ of a connected graph of order 12 that satisfies the hypotheses and has $\text{diam}(G) = 4$. This, of course, shows that the upper bound on diameter of the elementary connectivity argument really cannot be improved.

To show $G$ is connected, let $u$ and $v$ be vertices of $G$. If $u$ and $v$ are the same or adjacent, there is really nothing to do. So suppose that they are neither. There is a vertex, $w$, with $\deg(w) = \Delta(G)$. If either $u$ or $v$ is $w$, then we may proceed as in the proof of Theorem 2.4 to show there is a path with length at most 2 from $u$ to $v$. Thus, suppose that neither $u$ nor $v$ is $w$. Since $\deg(u) + \deg(w) \geq \delta(G) + \Delta(G) \geq n - 1$, there must be a vertex, $x$, in $G$ that is adjacent to both $u$ and $w$. Also, since $\deg(v) + \deg(w) \geq \delta(G) + \Delta(G) \geq n - 1$, there is a vertex, $y$, in $G$ that is adjacent to both $v$ and $w$. Plainly,
\[ W: u, x, w, y, v \]
is a walk of length 4 from $u$ to $v$. From Theorem 1.6, there is a path from $u$ to $v$ with length at most 4. Consequently $G$ must be connected with $\text{diam}(G) \leq 4$.

To see that the bound $n - 1$ is sharp, one need only consider the graph $G = 2K_k$ with $k \geq 3$. $G$ has order $n = 2k$, two components, and $\Delta(G) = \delta(G) = k - 1$. So $\Delta(G) + \delta(G) = 2k - 2 = n - 2$.

Next, let's consider the little matter of "$\text{diam}(G) \leq 2$". Let $H$ be the graph built from $G = 2K_k$ with $k \geq 3$ as follows: Choose a vertex from one of the $K_k$'s of $G$, and label it $u$. Then choose a vertex from the other $K_k$ and label it $v$. Add the edge $uv$. Then $H$ is connected, $\deg(u) = \deg(v) = k$, and if $x$ is a vertex different from $u$ and $v$, $\deg(x) = k - 1$. Thus,
\[ \Delta(H) + \delta(H) = k + (k - 1) = n - 1. \]
It is easy but slightlen tedious to verify that $\text{diam}(H) = 3$. [Draw a picture of the case $k = 3$!]
Finally, consider the connected graph $G$ below constructed by beginning with 3 $K_3$'s and then adding edges so as to obtain connectivity, but with enough care so that $\Delta(G) = \deg(c) = 8$, $\delta(G) = 3$, and $\text{diam}(G) = 4 = d(e,i) = d(e,k)$.

Can you find an example of smaller order??