1.5 Continuity

Definition

A function $f$ is said to be continuous at a point $c$ if the following 3 conditions are satisfied:

1) $f(c)$ is defined
2) $\lim_{x \to c} f(x)$ exists
3) $\lim_{x \to c} f(x) = f(c)$. 
1. $f(x) = \frac{1}{x-1}$ is continuous at any point except $c = 1$. The function does not satisfy condition 1) for example.
2. 

\[ f(x) = \begin{cases} \frac{1}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases} \]

This function is not continuous at \( c = 1 \) because condition 2) is not satisfied, it has no limit when \( x \to 1 \).
3. 

\[ f(x) = \begin{cases} 
2x + 1 & \text{if } x<2 \\
3x - 1 & \text{if } x>2 \\
2 & \text{if } x=2 
\end{cases} \]

Not continuous at \( c = 2 \) because 

\[ \lim_{x \to 2} f(x) \neq f(2). \]
Which functions are continuous and where!

. Polynomials are continuous everywhere.

. Rational functions are continuous wherever the denominators are not zero valued!

Example:

\[ f(x) = \frac{2x + 4}{x^2 + 2x + 1} \]

is not continuous at \( x = -1 \), but it is continuous anywhere else.
Theorem

If \( \lim_{x \to a} g(x) = L \) and if the function \( f \) is continuous at \( x = L \), then

\[
\lim_{x \to a} (f \circ g)(x) = f(L),
\]

that is to say:

\[
\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).
\]
Theorem

If the function $g$ is continuous at the point $c$ and the function $f$ is continuous at the point $g(c)$, then the function $f \circ g$ is continuous at $x = c$.

Proof: Check the three defining conditions for continuity!
The Intermediate Value Theorem

**Theorem (IVT)**

*If* \( f \) *is continuous on a closed interval* \([a, b]\) *and* \( c \) *is any number between* \( f(a) \) *and* \( f(b) \) *inclusive, then there is at least one number* \( x \) *in the interval* \([a, b]\) *such that* \( f(x) = c \).*
If $f$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval $(a, b)$. 
Approximate to within .01 (2 decimal accuracy or equivalently, absolute value of error less than .005) the solution of

\[ x - \cos x = 0 \]

in \([0, 1]\).

Consider \(f(x) = x - \cos x\). We need to approximate \(a\) in \([0, 1]\) such that \(f(a) = 0\). Notice that \(f(x)\) above satisfies the hypotheses of the IVT.
<table>
<thead>
<tr>
<th>$f(0)$</th>
<th>$f(1)$</th>
<th>Interval for $a$</th>
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<tbody>
<tr>
<td>$-1$</td>
<td>$.459$</td>
<td>$0 &lt; a &lt; 1$</td>
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<tr>
<td>$-37$</td>
<td>$.459$</td>
<td>$.5 &lt; a &lt; 1$</td>
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<tr>
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<td>$.7 &lt; a$</td>
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<tr>
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<td>$.7 &lt; a &lt; .8$</td>
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<td>$a &lt; .75$</td>
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<td>$.0015$</td>
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<td>$a &lt; .74$</td>
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<td>$.73 &lt; a &lt; .74$</td>
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<td>$-0068$</td>
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<td>$.735 &lt; a$</td>
</tr>
<tr>
<td>$-005$</td>
<td></td>
<td>$.736 &lt; a &lt; .74$</td>
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</tbody>
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1.6 Continuity of trigonometric, exponential and inverse functions

**Theorem**

*If f is invertible and continuous on its domain, then f\(^{-1}\) is continuous on its domain (the range of f).*

For \(b > 0, b \neq 1\), \(b^x\) and \(\log_b x\) are continuous on their domains (\(\mathbb{R}\) and \(\mathbb{R}_{0+}\) respectively.)