5.2 Properties of Rational functions

A **rational function** is a function of the form

\[ f(x) = \frac{\text{polynomial}}{\text{polynomial}} = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0} \]

**Example**

\[ f(x) = \frac{3x^4 - x^2 - 2x + 5}{-x^2 + 4x + 1} \]

The **domain** of a rational function is the set of all real numbers except those \( x \), for which \( q(x) = 0 \)

To **find the domain**:

(i) solve \( q(x) = 0 \)

(ii) Write \( Df = \{ x \mid q(x) \neq 0 \} \)

**Example:** Find the domain of \( f(x) = \frac{3x^4 - x^2 - 2x + 5}{-x^2 + 4x + 1} \)

(i) Solve: denominator = 0

\[-x^2 + 4x + 1 = 0\]

\[ x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(1)}}{2(-1)} = \frac{-4 \pm \sqrt{20}}{-2} = \frac{-4 \pm 2\sqrt{5}}{-2} = 2 \pm \sqrt{5} \]

(ii) \( Df = \{ x \mid x \neq 2 \pm \sqrt{5} \} = (-\infty, 2 - \sqrt{5}) \cup (2 - \sqrt{5}, 2 + \sqrt{5}) \cup (2 + \sqrt{5}, +\infty) \)

A rational function often has asymptotes: vertical and/or horizontal/oblique.

Informally speaking, an **asymptote** is a straight line (vertical, horizontal or slanted) toward which the graph comes near.

**How to find asymptotes**

**Vertical:** 1. Reduce \( f(x) \) to the lowest terms:

(i) factor completely the numerator and the denominator;

(ii) cancel common factors

2. Solve the equation: denominator = 0

3. If \( x = r \) is a solution found in 2, then the line \( x = r \) is a vertical asymptote

**Horizontal:**

a) if the degree of the numerator < the degree of the denominator, then the line \( y = 0 \) is the horizontal asymptote
(b) if the degree of the numerator = the degree of the denominator, then the line \( y = \frac{a_n}{b_k} \) is the horizontal asymptote

(c) if the degree of the numerator > the degree of the denominator, then the graph does not have a **horizontal asymptote**, however, if

**Oblique:**

(d) the degree of the numerator = 1 + the degree of the denominator, then the line \( y = (\text{quotient obtained by dividing the numerator by the denominator}) \) is an oblique (slanted) asymptote.

**Remarks:**

1. A rational function can have only one horizontal/oblique asymptote, but many vertical asymptotes.
2. If a rational function has a horizontal asymptote, then it does not have an oblique one.
3. The graph of a rational function can cross a horizontal/oblique asymptote, but does not cross a vertical asymptote.
4. Horizontal/oblique asymptotes describe the behavior of function for x with large absolute value; vertical asymptotes describe the behavior of function near a point.

**Example:** Find the asymptotes for the following functions

a) \( f(x) = \frac{3x + 5}{2x - 6} \)

**Vertical asymptote:**

1) \( f \) is in lowest terms
2) \( 2x - 6 = 0 \)
   - \( 2x = 6 \)
   - \( x = 3 \)
3) vertical asymptote: \( x = 3 \)

**Horizontal/oblique asymptote:**

degree of numerator (1) = degree of the denominator (1), \( y = \frac{3}{2} \) is the horizontal asymptote

b) \( f(x) = \frac{2x^2 + 5x - 1}{3x^3 - 6x^2} \)

**Vertical asymptote:**

1) \( f(x) = \frac{2x^2 + 5x - 1}{3x^3 - 6x^2} = \frac{2x^2 + 5x - 1}{3x^2(x - 2)} \) is in lowest terms (numerator can’t be factored)
2) \( 3x^2 - 6x^2 = 0 \)
   - \( 3x^2(x - 2) = 0 \)
   - \( x^2 = 0 \) or \( x - 2 = 0 \)
   - \( x = 0 \) or \( x = 2 \)
3) vertical asymptotes: \( x = 0, x = 2 \)

**Horizontal/oblique asymptote:**
degree of numerator (2) < degree of the denominator(3), \( y = 0 \) is the horizontal asymptote

c) \( f(x) = \frac{3x^5 - 1}{x^2 + 2} \)

*Vertical asymptote:* 1) \( f(x) \) is in lowest terms (the denominator cannot be factored)
   2) \( x^2 + 2 = 0 \)
      \( x^2 = -2 \) (not possible)
      no solution
   3) vertical asymptotes: none

*Horizontal/oblique asymptote:*  
degree of numerator (5) > degree of the denominator(2), there is no horizontal asymptote  

\[ \text{degree of numerator (5) } \neq 1 + \text{degree of the denominator(2), there is no oblique asymptote} \]

\[ d) \ f(x) = \frac{3x^3 - 4x^2 + 1}{x^2 - 2} \]

*Vertical asymptote:* 1) \( f(x) \) is in lowest terms
   2) \( x^2 - 2 = 0 \)
      \( x^2 = 2 \)
      \( x = \pm \sqrt{2} \)

   3) vertical asymptotes: \( x = -\sqrt{2}, x = \sqrt{2} \)

*Horizontal/oblique asymptote:* 
degree of numerator (3) > degree of the denominator(2), there is no horizontal asymptote 

\[
\begin{align*}
\frac{3x - 4}{x^2 - 2} &= \frac{3x^3 - 4x^2 + 1}{x^2 - 2} \\
&= \frac{-3x^3 + 6x}{-4x^2 + 6x + 1} \\
&= \frac{4x^2 - 8}{6x - 7}
\end{align*}
\]

Oblique asymptote: \( y = 3x - 4 \)
5.3 Sketching the graph of a rational function \( f(x) = \frac{p(x)}{q(x)} \)

1. Find the **domain**: (i) solve \( q(x) = 0 \)
   
   (ii) \( D_f = \{ x \mid q(x) \neq 0 \} \)

2. Find \( x \)- and \( y \)-intercepts: 
   - \( y \)-intercept: \( y = f(0) \)
   - \( x \)-intercepts: numerator = 0

3. Find **vertical asymptotes**, if any

   **Remark:** If \( x = r \) is excluded from the domain and \( x = r \) is not a vertical asymptote, then the graph of \( f \) will pass through the point \( (r, \text{“reduced”} f(r)) \) but the point itself will not be included. We put an open circle around that point.

   The graph of \( f \) has a “hole” at \( x = r \)

4. Find the **horizontal/oblique asymptote**, if any.

5. Find the points where the graph **crosses** the horizontal/oblique **asymptote** \( y = mx + b \)
   
   (i) solve the equation \( f(x) = mx + b \)

6. Check for **symmetries**

   (i) If \( f(-x) = f(x) \), then the graph is symmetric about \( y \)-axis;
   
   (ii) If \( f(-x) = -f(x) \), then the graph is symmetric about the origin

   **Remark:** If the graph is symmetric then only graph function for \( x > 0 \) and use symmetry to graph the corresponding part for \( x < 0 \)

7. Make the **sign chart** for the “reduced” \( f(x) \)

   (i) plot \( x \)-intercepts and points excluded from the domain on the number line; these points divide the number line into a finite number of test intervals
   
   (ii) choose a point in each test interval and compute the value of \( f \) at the test point
   
   (iii) based on the sign of \( f \) at the test point, assign the sign to each test interval

   **Remark:** When \( f(x) > 0 \), then the graph of \( f \) is above the \( x \)-axis.

   When \( f(x) < 0 \), then the graph is below the \( x \)-axis

8. Sketch the **graph** of \( f \) using 1)-7):

   (i) Draw coordinate system and draw all asymptotes using a dashed line
   
   (ii) plot the intercepts, points where the graph crosses the horizontal/oblique asymptote and the points from the table in step 7.
   
   (iii) join the points with a continuous curve taking into consideration position of the graph relative to the \( x \)-axis (step 7) and behavior near asymptotes.
Example: Graph \( f(x) = \frac{x^2 + x - 12}{x^2 - 4} \)

1) Domain: \( x^2 - 4 = 0 \)
   \( x^2 = 4 \)
   \( x = 2, x = -2 \)
   \( Df = \{x | x \neq -2, 2\} \)

2) \( y \)-intercept: \( y = f(0) = (-12)/(-4) = 3 \)

   \( x \)-intercepts: \( x^2 + x - 12 = 0 \)
   \( (x+4)(x-3) = 0 \)
   \( x = -4 \) or \( x = 3 \)

3) Vertical asymptotes: \( f(x) = \frac{x^2 + x - 12}{x^2 - 4} = \frac{(x+4)(x-3)}{(x-2)(x+2)} \)
   \( (x-2)(x+2) = 0 \)
   \( x = 2 \) \( x = -2 \)

   Vertical asymptotes: \( x = -2, x = 2 \)

4) Horizontal/oblique asymptotes
   Degree of numerator(2) = degree of denominator(2), \( y = 1/1 = 1 \) is the horizontal asymptote

5) Intersection with asymptote: \( f(x) = 1 \)
   \( \frac{x^2 + x - 12}{x^2 - 4} = 1 \)
   \( x^2 + x - 12 = x^2 - 4 \)
   \( x = 8 \)

   The graph crosses the horizontal asymptote at \( x = 8 \), that is at the point \((8, 1)\)

6) Symmetries:
   \( f(x) = \frac{x^2 + x - 12}{x^2 - 4} \)
   \( f(-x) = \frac{(-x)^2 + (-x) - 12}{(-x)^2 - 4} = \frac{x^2 - x - 12}{x^2 - 4} \)

   \( f(x) \) is not the same as \( f(-x) \), so \( f \) is not even and therefore not symmetric about \( y \)-axis

   \( f(-x) = \frac{(-x)^2 + (-x) - 12}{(-x)^2 - 4} = \frac{x^2 - x - 12}{x^2 - 4} \)

   \( -f(x) = -\frac{x^2 + x - 12}{x^2 - 4} = -\frac{x^2 - x + 12}{x^2 - 4} \)

   \( f(-x) \) and \( -f(x) \) are not the same so, \( f \) is not odd and therefore not symmetric about the origin

7)
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = \frac{x^2 + x - 12}{x^2 - 4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>$\frac{(-5)^2 + (-5) - 12}{(-5)^2 - 4} = \frac{8}{21}$ positive</td>
</tr>
<tr>
<td>-3</td>
<td>$\frac{(-3)^2 + (-3) - 12}{(-3)^2 - 4} = -\frac{6}{5}$ negative</td>
</tr>
<tr>
<td>0</td>
<td>3 positive</td>
</tr>
<tr>
<td>2.5</td>
<td>$\frac{2.5^2 + 2.5 - 12}{2.5^2 - 4} = \frac{13}{9}$ negative</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{4^2 + 4 - 12}{4^2 - 4} = \frac{2}{3}$ positive</td>
</tr>
</tbody>
</table>

8) 

\[ \begin{array}{c|c|c|c|c|c} 
\text{pos} & \text{neg} & \text{pos} & \text{neg} & \text{pos} \\
\hline 
-4 & -2 & 2 & 3 & \end{array} \] 

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