3.1 Operations on functions

Suppose two functions \( f \) and \( g \) are given and their domains are \( D_f \) and \( D_g \) respectively. We can form new functions using \( f \) and \( g \):

- **The sum**, \( f + g \), is the function defined as \( (f+g)(x) = f(x) + g(x) \)
- **The difference**, \( f - g \), is the function defined as \( (f-g)(x) = f(x) - g(x) \)
- **The product**, \( f \cdot g \), is a function defined as \( (f\cdot g)(x) = f(x)\cdot g(x) \)
- **The domain** of these functions is the intersection of the domains \( D_f \cap D_g \)

We can also form the **quotient**, \( f/g \), which is defined as \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \). The domain of \( f/g \) is the intersection of \( D_f \cap D_g \) except for those values \( x \) for which \( g(x) = 0 \). We will write this as

\[
D(f/g) = \{ x | x \in D_f \cap D_g \text{ and } g(x) \neq 0 \}
\]

Example: Suppose that \( f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{x} \). Find a) \( (f-g)(4) \), b) \( (f/g)(x) \) and its domain.

a) \( (f-g)(4) = f(4) - g(4) = \sqrt{4} - \frac{1}{4} = 2 - \frac{1}{4} = \frac{7}{4} \)

b) \[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{1} = x\sqrt{x}
\]

\( D_f = [0, +\infty) \), \( D_g = \{ x | x \neq 0 \} \). \( D_f \cap D_g = (0, +\infty) \), since \( g(x) = 0 \) has no solutions, there is nothing to exclude from \( D_f \cap D_g \) and \( D(f/g) = D_f \cap D_g = (0, +\infty) \)
6.1 Composition of function

The composition of \( f \) and \( g \), denoted \( f \circ g \), and read \( g \) composed with \( f \) or “f circle \( g \)”, is a function defined as

\[
(f \circ g)(x) = f(g(x))
\]

Composition is an action of two functions in a succession. If \( x \) is an input, then \( g \) acts on \( x \) first, producing the output \( g(x) \), which becomes an input for \( f \) and produces the output \( f(g(x)) \).

The domain of \( f \circ g \) is the set of all values \( x \) that are in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \).

We will write \( D(f \circ g) = \{x \mid x \in Dg \text{ and } g(x) \in Df \} \)

**Example:** Find the composition \( f \circ g \) and its domain, if \( f(x) = \frac{2x+3}{x-4} \) and \( g(x) = \frac{2}{x} \)

To find \((f \circ g)(x)\), just replace every \( x \) in the formula for \( f(x) \) by the formula for \( g(x) \)

\[
(f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x}\right) = \frac{\left(\frac{2}{x}\right) + 3}{\left(\frac{2}{x}\right) - 4} = \frac{\frac{4 + 3}{x} \cdot x}{2 - 4 \cdot \frac{2}{x}} = \frac{4 + 3x}{2 - 4x}
\]

To find the domain:

1) Find the domain of \( g \) and then and
   \( Dg = \{x \mid x \neq 0\} \)

2) Find the domain of \( f \)
   \( Df = \{x \mid x \neq 4\} \)

3) Find \( x \)'s for which \( g(x) \) is in the domain of \( f \)
   \( g(x) = \frac{2}{x} \)

   we want \( g(x) \) to be in the domain of \( f \), that is we want \( g(x) \neq 4 \).

   \( \frac{2}{x} \neq 4 \) means \( 2 \neq 4x \) or \( x \neq 1/2 \)

4) Find the intersection (common part) of sets found in 1) and 3)
   \( D(f \circ g) = \{x \mid x \neq 0, \frac{1}{2}\} \)

**Another way to find the domain** is to find the domain of the expression obtained immediately after substituting \( g \) into \( f \). In the example, that expression is
Determine the domain of this expression.

Here x cannot be zero \((x \neq 0)\), because it would make the “small” fractions undefined. But also, the main denominator \(\frac{2}{x} - 4\) cannot be zero. When is it zero?

\[
\frac{2}{x} - 4 = 0
\]

\[
\frac{2}{x} = 4
\]

\[
4x = 2
\]

\[
x = \frac{1}{2}
\]

Therefore x must be different than \(\frac{1}{2}\) \((x \neq \frac{1}{2})\).

Altogether \(D(f \circ g) = \{x | x \neq 0, \frac{1}{2}\}\).

Often, it is useful to find two functions whose composition yields a given function.

**Example:** Find two functions \(f\) and \(g\) so that \((f \circ g)(x) = \sqrt{x + 4}\).

We are looking for two functions \(f\) and \(g\) such that \(f(g(x)) = \sqrt{x + 4}\)

Think in the following way: starting with any value of \(x\) what **two** actions have to be applied to \(x\) in order to obtain the given value \(\sqrt{x + 4}\).

In this example, starting with \(x\) we must compute \(x + 4\) first \((1^{st}\) action) and then take the square root \((2^{nd}\) action).

Therefore \(g(x) = x + 4\) \((1^{st}\) action) and \(f(x) = \sqrt{x}\) \((2^{nd}\) action)

It is often helpful to read the formula out loud and translate the phrase into formulas

(i) **Read:** the square root of \(x\) plus 4.

(ii) Divide the phrase into two groups, one before the word of \(g\) (the square root) and one after \((x\) plus 4). The two groups correspond to two functions, \(f\) and \(g\), respectively.

Translate the phrases into formulas

\(f\): The square root --- \(f(x) = \sqrt{x}\)

\(g\): \(x\) plus 4 ---- \(g(x) = x + 4\)

This method works perfectly for all functions except the powers, like \((x + 4)^{3}\). But, if you learn to read this expression as “the third power of \(x\) plus 4” rather than “\(x\) plus 4 to the third power”, you will have no problem with this function either.

**Example:** Find two functions \(f\) and \(g\) so that \((f \circ g)(x) = h(x)\), where \(h(x) = |x^2 + 4|\). None of the functions can be the identity.

(i) **Read:** absolute value of \(x\) squared plus 4

(ii) Divide absolute value of \(x\) squared plus 4

(iii) Define \(f(x) = |x|\), \(g(x) = x^2 + 4\)
6.2 One to one functions; Inverse functions

One to one functions
A function is one to one if any two different inputs (x’s) correspond to two different outputs (y’s).
That is if \( x_1 \neq x_2 \) then \( f(x_1) \neq f(x_2) \)
Example

![Diagram of a one to one function](image)

Function, but **not one to one** since different inputs (2 and 3) correspond to the same output

![Diagram of a function that is not one to one](image)

**One to one** function- no two inputs correspond to the same output

It is easy to recognize a one to one function if its graph is given. A one to one function will pass the Horizontal Line Test

**Horizontal Line Test:** If every horizontal line intersects the graph of a function \( f \) at most at one point, then \( f \) is one to one

*Example:* Which of the following graphs represent a one to one function?

![Graphs](image)

*Not a one to one*  
*one to one*  
*not a function*
Inverse functions

If a function $f$ is one to one it has the inverse, denoted by $f^{-1}$. If a function $f$ assigns to an $x$ a $y$ (that is $f(x) = y$), then the inverse $f^{-1}$ will assign to that $y$ the element $x$ ($f^{-1}(y) = x$).

**Example:**

Function $f = \{(1,a), (2,b), (3,c)\}$
Function $f^{-1} = \{(a,1), (b,2), (c,3)\}$

If $(x,y)$ belongs to $f$ then $(y,x)$ belongs to $f^{-1}$

Here are the properties of the inverse functions

1. If $y = f(x)$ then $x = f^{-1}(y)$ or equivalently if $x = f(y)$, then $y = f^{-1}(x)$
2. Domain of $f^{-1} =$ range of $f$
   Range of $f^{-1} =$ domain of $f$
3. $(f o f^{-1})(x) = x$ for all $x$ in the domain of $f^{-1}$
   $(f^{-1} o f)(x) = x$ for all $x$ in the domain of $f$
   $(f^{-1}$ undoes what $f$ did )
4. The graphs of $f$ and $f^{-1}$ are symmetric with respect to the line $y = x$.
   If $(x, y)$ is a point on the graph of $f$, then $(y,x)$ is on the graph of $f^{-1}$. Points $(x,y)$ and $(y,x)$ are symmetric with respect to the line $y = x$

To check that two functions $f$ and $g$ are inverses of each other, we check that $f(g(x)) = x$ and $g(f(x)) = x$

**How to find the inverse** of a one to one function:

(i) Write the equation $y = f(x)$
(ii) Switch $x$ with $y$ to write $x = f(y)$
(iii) Solve the equation in (ii) for $y$. The solution is $y = f^{-1}(x)$

**Example:** Find the inverse of $f(x) = \frac{2x+3}{x-4}$ and its domain and range

(i) $y = \frac{2x+3}{x-4}$
(ii) $x = \frac{2y+3}{y-4}$
(iii) $x(y-4) = 2y + 3$
    $xy - 4x = 2y + 3$
\begin{align*}
xy - 2y &= 3 + 4x \\
y(x-2) &= 3 + 4x \\
y &= \frac{4x + 3}{x - 2} \\
f^{-1}(x) &= \frac{4x + 3}{x - 2}
\end{align*}

To find the domain and the range we use property 2 of inverse functions.

**Domain of** $f^{-1} = \{x | x \neq 2\} = (-\infty, 2) \cup (2, +\infty)$ = Range of $f$

**Domain of** $f = \{x | x \neq 4\} = (-\infty, 4) \cup (4, +\infty)$ = Range of $f^{-1}$

**Example:** Given the graph of a one to one function $f$, graph its inverse $f^{-1}$.

The graphs of $f$ and $f^{-1}$ are symmetric about the line $y = x$. Use the property that if $(x,y)$ is on the graph of $f$, then $(y,x)$ is on the graph of $f^{-1}$. Plot those points and draw the graph remembering the symmetry.

Points on the graph of $f$: (-2, -8), (-1, -1), (0, 0), (1,1), (2,8)
Points on the graph of $f^{-1}$: (-8,-2), (-1,-1), (0,0), (1,1), (8,2)
6.3 Exponential Functions

A function of the form \( f(x) = a^x, \ a > 0, \ a \neq 1 \) is called an **exponential** function. Its domain is the set of all real numbers. For an exponential function \( f \) we have \( \frac{f(x+1)}{f(x)} = a \). The graph of an exponential function depends on the value of \( a \).

\[
\begin{align*}
\text{a} &> 1 \\
\text{0 < a < 1}
\end{align*}
\]

Points on the graph: \((-1, 1/a), (0,1), (1, a)\)

**Properties of exponential functions**

1. The domain is the set of all real numbers: \( \text{Df} = \mathbb{R} \)
2. The range is the set of positive numbers: \( \text{Rf} = (0, +\infty) \).
   
   This means that \( a^x \text{ is always positive} \), that is \( a^x > 0 \) for all \( x \). The equation \( a^x = \text{negative number} \) has no solution
3. There are no \( x \)-intercepts
4. The \( y \)-intercept is \((0, 1)\)
5. The \( x \)-axis (line \( y = 0 \)) is a **horizontal asymptote**
6. An exponential function is **increasing** when \( a > 1 \) and **decreasing** when \( 0 < a < 1 \)
7. An exponential function is one to one, and therefore has the inverse. The inverse of the exponential function \( f(x) = a^x \) is a logarithmic function \( g(x) = \log_a(x) \)
8. Since an exponential function is **one to one** we have the following property:

   If \( a^u = a^v \), then \( u = v \).

   This property is used when solving exponential equations that could be rewritten in the form \( a^u = a^v \).

**Natural exponential function** is the function \( f(x) = e^x \), where \( e \) is an irrational number, \( e \approx 2.718281\ldots \)

The number \( e \) is defined as the number to which the expression \( (1 + \frac{1}{n})^n \) approaches as \( n \) becomes larger and larger. Since \( e > 1 \), the graph of the natural exponential function is as below.
Example: Use transformations to graph \( f(x) = 3^{-x} - 2 \). Start with a basic function and use one transformation at a time. Show all intermediate graphs.
This function is obtained from the graph of \( y = 3^x \) by first reflecting it about y-axis (obtaining \( y = 3^{-x} \)) and then shifting the graph down by 2 units. Make sure to plot the three points on the graph of the basic function!

Remark: Function \( y = 3^x \) has a horizontal asymptote, so remember to shift it too when performing shift up/down.

Example: Solve \( 4^{x^2} = 2^x \)

(i) Rewrite the equation in the form \( a^u = a^v \).
Since \( 4 = 2^2 \), we can rewrite the equation as
\[
(2^2)^{x^2} = 2^x
\]
Using properties of exponents we get \( 2^{2x^2} = 2^x \).

(ii) Use property 8 of exponential functions to conclude that \( u = v \).
Since \( 2^{2x^2} = 2^x \) we have \( 2x^2 = x \).

(iii) Solve the equation \( u = v \)
\[
2x^2 = x \\
2x^2 - x = 0 \\
x(2x - 1) = 0 \\
x = 0 \quad 2x - 1 = 0 \\
x = 1/2 \]
Solution set = \{0, \frac{1}{2}\}

6.4 Logarithmic functions

A \textbf{logarithmic} function \( f(x) = \log_a(x) , \ a > 0, \ a \neq 1, \ x > 0 \) (logarithm to the base \( a \) of \( x \)) is the inverse of the exponential function \( y = a^x \).

Therefore, we have the following properties for this function (as the inverse function)

\[(I) \quad y = \log_a(x) \text{ if and only if } a^y = x \]

This relationship gives the definition of \( \log_a(x) \): \( \log_a(x) \) is \textit{an exponent to which the base \( a \) must be raised to obtain \( x \)}

\textbf{Example:}

a) \( \log_2(8) \) is an exponent to which 2 must be raised to obtain 8 (we can write this as \( 2^x = 8 \) ) Clearly this exponent is 3, thus \( \log_2(8) = 3 \)

b) \( \log_{1/3}(9) \) is an exponent to which \( 1/3 \) must be raised to obtain 9: \( (1/3)^x = 9 \). Solving this equation for \( x \), we get \( 3^{-x} = 3^2 \) and \( -x = 2 \) or \( x = -2 \). Thus \( \log_{1/3}(9) = -2 \).

c) \( \log_2(3) \) is an exponent to which 2 must be raised to obtain 3: \( 2^x = 3 \). We know that such a number \( x \) exists, since 3 is in the range of the exponential function \( y = 2^x \) (there is a point with \( y \)-coordinate 3 on the graph of this function) but we are not able to find it using traditional methods. If we want to refer to this number, we use \( \log_2(3) \).

The relationship in (I) allows us to move from exponent to logarithm and vice versa

\textbf{Example:}

- Change the given logarithmic expression into exponential form: \( \log_2 x = 4 \)
  The exponential form is: \( 2^4 = x \).
  Notice that this process allowed us to find value of \( x \), or to solve the equation \( \log_2(x) = 4 \)
- Change the given exponential form to the logarithmic one: \( 2^x = 3 \). Since \( x \) is the exponent to which 2 is raised to get 3, we have \( x = \log_2(3) \).
  Note that the base of the exponent is always the same as the base of the logarithm.

\textbf{Common logarithm} is the logarithm with the base 10. Customarily, the base 10 is omitted when writing this logarithm:

\( \log_{10}(x) = \log(x) \)

\textbf{Natural logarithm} is the logarithm with the base \( e \) (the inverse of \( y = e^x \)): \( \ln(x) = \log_e(x) \)

\[(II) \quad \text{Domain of a logarithmic function } = (0, \infty) \]
We can take a logarithm of a positive number only.

\[(III) \quad \log_a(a^x) = x, \text{ for all real number} \]
\( a^{\log_a(x)} = x, \text{ for all } x > 0 \)

\textbf{Example} \( \log_2 2^5 = 5, \quad \ln e^3 = 3, \quad 3^{\log_3(2)} = 2, \quad e^{\ln 7} = 7 \)

\[(IV) \quad \text{Graph of } f(x) = \log_a(x) \text{ is symmetric to the graph of } y = a^x \text{ about the line } y = x \]
Points on the graph of $y = \log_a(x)$: $(1/a, -1)$, $(1,0)$, $(a, 1)$

(V) The x-intercept is $(1, 0)$.
(VI) There is no y-intercept.
(VII) The y-axis (the line $x = 0$) is the vertical asymptote.
(VIII) A logarithmic function is increasing when $a > 1$ and decreasing when $0 < a < 1$.
(IX) A logarithmic function is one to one. Its inverse is the exponential function.
(X) Because a logarithmic function is one to one we have the following property:
   If $\log_a(u) = \log_a(v)$, then $u = v$.
   This property is used to solve logarithmic equations that can be rewritten in the form $\log_a(u) = \log_a(v)$.

Example: Use transformations to graph $f(x) = -2\log_3(x-1) + 3$. Start with a basic function and use one transformation at a time. Show all intermediate graphs. Plot the three points on the graph of the basic function.

a) $y = \log_3(x)$

b) $y = \log_3(x-1)$

c) $y = 2\log_3(x-1)$
d) \( y = -2\log_3(x-1) \)

e) \( y = -2\log_3(x-1) + 3 \)

*Remark:* Since a logarithmic function has a vertical asymptote, do not forget to shift it when shifting left/right

*Example:* Find the domain of the following functions (A logarithm is defined only for positive (> 0) values)

a) \( f(x) = \log_{1/2}(x^2 - 3) \)

\[ \text{Df: } x^2 - 3 > 0 \]
\[ x^2 - 3 = 0 \]
\[ x^2 = 3 \]
\[ x = \pm \sqrt{3} \]

\[ \text{Df} = (-\infty, -\sqrt{3}) \cup (\sqrt{3}, +\infty) \]

b) \( g(x) = \ln\left(\frac{2x + 3}{x^2 - 9}\right) \)

\[ \text{Dg: } \frac{2x + 3}{x^2 - 9} > 0 \]
\[ 2x + 3 = 0 \quad x^2 - 9 = 0 \]
\[ 2x = -3 \quad x^2 = 9 \]
\[ x = -3/2 \quad x = \pm 3 \]

Use the test points to determine the sign in each interval

\[ \text{Dg} = (-3, -3/2) \cup (3, +\infty) \]
Example: Solve the following equations

a) \( \log_5(x^2 + x + 4) = 2 \)

(i) Find the domain of the logarithm(s)
\[
x^2 + x + 4 > 0
\]
\[
x^2 + x + 4 = 0
\]
\[
x = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2} = \frac{-1 \pm \sqrt{-15}}{2} \text{ not a real number}
\]
Since \( y = x^2 + x + 4 \) has no x-intercepts and the graph is a parabola that opens up, the graph must always stay above x-axis. Therefore, \( x^2 + x + 4 > 0 \) for all x

(ii) Change the equation to the exponential form and solve
\[
x^2 + x + 4 = 5^2
\]
\[
x^2 + x + 4 = 25
\]
\[
x^2 + x - 21 = 0
\]
\[
x = \frac{-1 \pm \sqrt{1 - 4(1)(-21)}}{2} = \frac{-1 \pm \sqrt{85}}{2}
\]
since there are no restrictions on x, above numbers are solutions of the equation.

b) \( e^{-2x+1} = 13 \)
This is an exponential equation that can be solved by changing it to the logarithmic form
\[-2x + 1 = \log_e(13)\]
\[-2x + 1 = \ln(13)\]
\[-2x = -1 + \ln13\]
\[x = \frac{-1 + \ln13}{-2} = \frac{1 - \ln13}{2}\]
Since this is an exponential equations, there are no restrictions on x. Solution is \( x = \frac{1 - \ln13}{2} \)

6.4 Properties of logarithms

Properties of logarithms:

Suppose \( a > 0, a \neq 1 \) and \( M, N > 0 \)

(i) \( \log_a(1) = 0 \quad \log_a(a) = 1 \)

Example: \( \log_2(1) = 0 \quad \log_{15}(15) = 1 \)
\( \ln(1) = 0 \quad \ln(e) = 1 \)

(ii) \( a^{\log_a(M)} = M \)

Example: \( 6^{\log_6(7)} = 7 \quad e^{\ln(4)} = 4 \)

(iii) \( \log_a(a^r) = r \)

Example: \( \log_3(3^4) = 4 \quad \ln(e^{2x}) = 2x \)

(iv) \( \log_a(M \cdot N) = \log_a(M) + \log_a(N) \)
\( \log_a(M) + \log_a(N) = \log_a(M \cdot N) \)

Example: \( \log_5(10) = \log_5(5) + \log_5(2) \)
\( \ln(x+1) + \ln(x-1) = \ln[(x+1)(x-1)] \)

(v)
\[
\log_a \left( \frac{M}{N} \right) = \log_a (M) - \log_a (N)
\]

\[
\log_a (M) - \log_a (N) = \log_a \left( \frac{M}{N} \right)
\]

\[
\log_a \left( \frac{15}{2} \right) = \log_4(15) - \log_4(2)
\]

\[
\log_4 (12) - \log_4 (3) = \log_4 \left( \frac{12}{3} \right)
\]

---

(vi) \( \log_a(M^r) = r \cdot \log_a(M) \)

\[ r \cdot \log_a(M) = \log_a(M^r) \]

Example: \( \log(3^x) = x \log(3) \)

5\log_3(x+1) = \log_3 [(x+1)^5] \]

---

(vii) If \( M = N \), then \( \log_a(M) = \log_a(N) \)

If \( \log_a(M) = \log_a(N) \), then \( M = N \)

2x-5

---

(viii) Change of the base formula

\[
\log_a (M) = \frac{\log_b (M)}{\log_b (a)} , \quad \text{where } b \text{ is any positive number different than 1}
\]

In particular,

\[
\log_a (M) = \frac{\log(M)}{\log(a)} \quad \text{and} \quad \log_a (M) = \frac{\ln(M)}{\ln(a)}
\]

This formula is used to find values of logarithms using a calculator.

Example: Evaluate \( \log_2(3) \)

\[
\log_2(3) = \frac{\ln(3)}{\ln(2)} \approx 1.5849
\]

Example: Write \( \log_3 \left( \frac{x(x+2)^3}{\sqrt{x^2+1}} \right) \) as a sum/difference of logarithms. Express powers as product.

\[
\log_3 \left( \frac{x(x+2)^3}{\sqrt{x^2+1}} \right) = \log_3[x(x+2)^3] - \log_3(\sqrt{x^2+1}) = \\
\log_3(x) + 3\log_3(x+2) - \frac{1}{2} \log_3(x^2 + 1)
\]

Example: Write as a single logarithm

\[
3\log_4(3x+1) - 2\log_4(2x-1)- \log_4(x) = \\
= \log_4 [(3x+1)^3] - \log_4((2x-1)^3) - \log_4(x) = \\
= \log_4 \left( \frac{(3x+1)^3}{(2x-1)^2} \right) - \log_4(x) = \log_4 \left[ \frac{(3x+1)^3}{x(2x-1)^2} \right]
\]
6. 5 Exponential and logarithmic equations

A logarithmic equation is an equation that contains a variable “inside” a logarithm. Since a logarithm is defined only for positive numbers, before solving a logarithmic equation you must find its domain (alternatively, you can check the apparent solutions by plugging them into the original equation and checking whether all logarithms are well defined).

There are two types of logarithmic equations:

(A) Equations reducible to the form \( \log_a(u) = r \), where \( u \) is an expression that contains a variable and \( r \) is a real number

To solve such equation change it to the exponential form \( a^r = u \) and solve.

Example: Solve \( 3\log_2(x-1) + \log_2(3) = 5 \)

(i) Determine the domain of the equation. (What is “inside” of any logarithm must be positive)
\[ x-1 > 0 \]
\[ x > 1 \]
(Only numbers greater than 1 can be solutions of this equation)

(ii) Use properties of logarithms to write the left hand side as a single logarithm
\[ \log_2(x-1)^3 + \log_2(3) = 5 \]
\[ \log_2(3(x-1)^3) = 5 \]

(iii) Change to the exponential form
\[ 2^5 = 3(x-1)^3 \]

(iv) Solve
\[ 32 = 3(x-1)^3 \]
\[ 32/3 = (x-1)^3 \]
\[ x-1 = \sqrt[3]{32/3} \]
\[ x = 1 + \sqrt[3]{32/3} \]

(v) Since \( x = 1 + \sqrt[3]{32/3} \) is greater than 1, it is the solution

(B) Equations reducible to the form \( \log_a(u) = \log_a(v) \).

To solve such equation use the (vii) property of logarithms to get the equation \( u = v \). Solve the equation.

Example: Solve \( \log_5(x) + \log_5(x-2)= \log_5 (x+ 4) \).

(i) Determine the domain of the equation. (What is “inside” of any logarithm must be positive)
\[ x > 0 \] and \( x - 2 > 0 \) and \( x + 4 > 0 \)
\[ x > 0 \] and \( x > 2 \) and \( x > -4 \)
If \( x \) is to satisfy all these inequalities, then \( x > 2 \)
(Only numbers greater than 2 can be solutions of this equation)
(ii) Use properties of logarithms to write each side of the equation as a single logarithm

\[ \log_5(x(x-2)) = \log_5(x + 4) \]

(iii) Since the logarithms are equal (\( \log_a(M) = \log_a(N) \)), we must have (\( M = N \))

\[ x(x-2) = x + 4 \]

(iv) Solve

\[ x(x-2) = x + 4 \]
\[ x^2 - 2x = x + 4 \]
\[ x^2 - 3x - 4 = 0 \]
\[ (x-4)(x+1) = 0 \]
\[ x = 4 \quad \text{or} \quad x = -1 \]

(v) Since any solution must be greater than 2, only \( x = 4 \) is the solution

Exponential equations

These are equations in which a variable appears in the exponent. Since exponential functions are defined for all real numbers, there are no restrictions on a variable and we do not have to check the solutions.

There are three types of exponential equations:

(A) **Equations that can be reduced to the form** \( a^u = r \), where \( u \) is an expression that contains a variable and \( r \) is a positive real number. If \( r \) is negative or 0, the equation has no solution.

**To solve** such equation, change into logarithmic form and solve

*Example:* Solve \( 3 \cdot 4^{2x-1} = 5 \)

(i) Write the equation in the desired form (exponent = a number)

\[ 4^{2x-1} = \frac{5}{3} \]

(ii) Change to the logarithmic form

\[ 2x-1 = \log_4(\frac{5}{3}) \]

(iii) Solve

\[ 2x = 1 + \log_4(\frac{5}{3}) \]
\[ x = \frac{1 + \log_4(\frac{5}{3})}{2} \]

To find an approximate value, use the change of the base formula to rewrite \( \log_4(\frac{5}{3}) \) as \( \frac{\log(\frac{5}{3})}{\log 4} \)

(B) **Equations that can be reduced to the form** \( a^u = a^v \).

**To solve** such an equation use the property of exponential functions that says that if \( a^u = a^v \), then \( u = v \) and solve it.

*Example* Solve \( (16)^y \cdot 2^x = 4^6 \)

(i) Use the properties of exponents to write the equation in the desired form. Notice that all bases \( (16, 2, 4) \) are powers of 2, \( 16 = 2^4 \), \( 2 = 2^1 \), \( 4 = 2^2 \).

\[ (16)^y \cdot 2^x = 4^6 \]
\((2^4)^x \cdot 2^{x^2} = (2^2)^6\)
\[2^{4x} \cdot 2^{x^2} = 2^{12}\]
\[2^{4x + x^2} = 2^{12}\]

(ii) Use the property (7)
\[4x + x^2 = 12\]

(iii) Solve
\[x^2 + 4x - 12 = 0\]
\[(x+6)(x-2) = 0\]
\[x = -6 \text{ or } x = 2\]

Solutions: -6, 2

(C) Equations that can be reduced to the form \(a^u = b^v\)

To solve such equation apply the log (or \(\ln\)) to both sides of the equation (property (vii) of logarithms), use the property of logarithms to bring the \(u\) and \(v\) outside of the logarithms and solve for the variable. Keep in mind that \(\log(a)\) and \(\log(b)\) are just numbers (like 1.34 or 3).

Example: Solve \(2^{x+1} = 5^{-1 - 2x}\)

(i) Apply \(\log\) to both sides
\[\log(2^{x+1}) = \log(5^{-1 - 2x})\]

(ii) Use properties of logarithms. (Enclose the powers into the parentheses)
\[(x+1)\log(2) = (1 - 2x)\log(5)\]

(iii) Solve
Eliminate parentheses
\[x\log(2) + \log(2) = \log(5) - 2x\log(5)\]
Bring the terms with \(x\) to the left hand side
\[x\log(2) + 2x\log(5) = \log(5) - \log(2)\]
Factor out \(x\)
\[x(\log(2) + 2\log(5)) = \log(5) - \log(2)\]
Divide, to find \(x\)
\[x = \frac{\log(5) - \log(2)}{\log(2) + 2\log(5)}\]
You could use properties of logarithms to write the solution as
\[x = \frac{\log(5/2)}{\log(2 \cdot 5^2)} = \frac{\log(5/2)}{\log(50)}\]

If an exponential equation cannot be transformed to one of the types above, try to substitute by \(u\) an exponential expression within the equation. This might reduce the equation to an algebraic one, like quadratic or rational.

Example: Solve \(2^{2x} + 2^{x+2} - 12 = 0\)

(i) Rewrite the equation so that \(2^x\) appears explicitly
\[(2^x)^2 + 2^x \cdot 2^2 - 12 = 0\]
\[(2^x)^2 + 4(2^x) - 12 = 0\]

(ii) Substitute \(u = 2^x\)
\[u^2 + 4u - 12 = 0\]

(iii) Solve the equation for \(u\)
(u+6)(u-2) = 0
u = -6 or u = 2
(iv) Back-substitute and solve for x
2^x = -6 or 2^x = 2
No solution x = 1
Solution: x = 1

6.7 Compound Interest

**Interest** (I): money paid for the use of money

**Principal** (P): money invested in a bank or money borrowed from the bank

**Interest rate** (r): amount (as a percent) charged for the use of the principal for a given period of time (usually a year, that is *per annum*)

If a principal P is invested for t years at a per annum interest rate r, expressed as a decimal, then the **simple interest** earned is \( I = Prt \)

Simple interest is computed once at the end of the payment period.

**Remark:** \( r \) is always expressed as decimal and \( t \) is always in years

The interest can be computed certain number of times during a year and each time added to the principal, so the next time the interest is computed on the old principal + added interest. We say then that the interest is **compounded**. The interest can be computed
- twice per year, in which case we say it is compounded semiannually
- 4 times per year, in which case we say it is compounded quarterly
- 12 times per year, in which case we say it is compounded monthly
- 365 times per year, in which case we say it is compounded daily

**Example:** Suppose that the interest is compounded quarterly, that is, the interest is computed (and added to the principal) 4 times per year. If on Jan 1, principal P is invested at the per annum rate r, then on April 1 (first quarter; \( t = \frac{1}{4} \)) the interest is computed and is \( I = Pr(1/4) \). That interest is added to the original principal P, so the new principal is \( P_1 = P + I = P + Pr(1/4) = P(1+r/4) \).

This new principal is re-invested for another quarter of the year (\( t = \frac{1}{4} \)) and on July 1, the interest is computed again (\( I = P_1r(1/4) \)) and added to the principal \( P_1 \). Thus, after two quarters, the total amount in the bank is \( P_2 = P_1 + I = P_1 + P_1r(1/4) = P_1(1 + r/4) = P_1(1 + r/4)(1 + r/4) = P_1(1 + r/4)^2 \).

\( P_2 \) becomes the new principal and the process is repeated.

After n quarters, the amount in the bank is \( A = P \left( 1 + \frac{r}{4} \right)^n \), so after t years (4t quarters), the amount is

\[ A = P \left( 1 + \frac{r}{4} \right)^{4t} \]

In general:

If a principal P is invested for t years at an annual interest rate r (in decimal form) **compounded n times** per year, then the amount A in the account is

\[ A = P \left( 1 + \frac{r}{n} \right)^{nt} \]

A is often called *future value of the investment* and P the *present value*.
Example: Find the amount that results from investing $300 at 12% compounded monthly after the period of 18 months.

Given:
\( P = 300 \)
\( r = 12\% = 0.12 \)
\( t = 18 \text{ months} = 1.5 \text{ years} \)
\( n = 12 \)

\[
A = P \left(1 + \frac{r}{n}\right)^{nt} = 300 \left(1 + \frac{0.12}{12}\right)^{12 \times 1.5} \approx 358.84
\]

Example: Find the principal needed (or the present value) to get $75 after 3 years at 8% compounded quarterly.

Given:
\( A = 75 \)
\( t = 3 \)
\( r = 8\% = 0.08 \)
\( n = 4 \)

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]
\[
75 = P \left(1 + \frac{0.08}{4}\right)^{4 \times 3}
\]
\[
75 = P(1.02)^{12}
\]
\[
P = \frac{75}{(1.02)^{12}} = 75(1.02)^{-12} \approx 59.14
\]

Example: Find the amount that results from investing $1000 at 10% compounded annually, semiannually, quarterly, monthly, daily, hourly and every minute after the period of one year.

Given:
\( P = 1000 \)
\( r = 10\% = 0.1 \)
\( t = 1 \)

\( n = 1: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{1}\right)^1 = 1100.00 \)

\( n = 2: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{2}\right)^2 \approx 1102.50 \)

\( n = 4: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{4}\right)^4 \approx 1103.81 \)

\( n = 12: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{12}\right)^{12} \approx 1104.71 \)

\( n = 365: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{365}\right)^{365} \approx 1105.15 \)

\( n = 8760: \quad A = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.1}{8760}\right)^{8760} \approx 1105.17 \)
n=525600: \[ A = P\left(1+\frac{r}{n}\right)^{nt} = 1000\left(1+\frac{0.1}{525600}\right)^{525600} \approx $1105.17 \]

As the example shows, more frequent compounding results in the larger accumulation, but there is a limit to how much can be earned. That limiting value is obtained by using what we call continuous compounding.

If a principal \( P \) is invested for \( t \) years at an annual interest rate \( r \) (in decimal form) **compounded continuously**, then the amount \( A \) in the account is

\[ A = Pe^{rt} \]

**Example:** If \( P = $1000, r = 10\% = 0.1 \) and \( t= 1 \), we get \( A = Pe^{rt} = 1000e^{0.1} \approx $1105.17 \)

(Compare with the amounts in the previous example)

**Example:** If Angela has $100 to invest at 6\% per annum **compounded continuously**, how long will it be before she has $175?

**Given:**
- \( P = 100 \)
- \( r = 6\% = 0.06 \)
- \( A = 175 \)

\[ A = Pe^{rt} \]
\[ 175 = 100e^{0.06t} \]
\[ 175/100=e^{0.06t} \]
\[ 0.06t = \ln(1.75) \]
\[ t = \ln(1.75)/0.06 \approx 9.327 \text{ years} = 9 \text{ years} 3 \text{ months} 28 \text{ days} \]

---

6.8 Exponential Growth and Decay Models

We say that a quantity \( N \) increases (grows) exponentially with time \( t \) if \( N(t) = N_0e^{rt} \), where \( r \) is a positive constant, called the growth rate, and \( N_0 \) is the initial value of \( N \) at time \( t = 0 \). (Time \( t \) can be measured in different units: years, weeks, days, etc.)

In many natural situations, the growth of populations is exponential under assumption that food supply and space are unlimited. Such models, expressed by the formula \( N(t) = N_0e^{rt} \) are said to follow the **law of uninhibited growth**.

The graph of such exponential growth is below
Example: The population of a Midwestern city follows the exponential law.

a) If N is the population of the city and t is the time in years, express N as a function of t.
   \[ N = N_0 e^{rt} \]

b) If the population decreased from 900,000 to 800,000 from 2003 to 2005, what will the population be in 2007?

   In 2003, the population was 900,000. So, assuming that 2003 is the initial year (t = 0) we have
   \[ N_0 = 900,000 \] and hence \[ N(t) = 900,000 e^{rt}. \]

   In 2005 (t = 2), the population was 800,000. That is
   \[ 800,000 = 900,000 e^{2r}. \]
   This equation has only one variable, r. Solve, to find it.
   \[ e^{2r} = \frac{8}{9} \]
   \[ 2r = \ln\left(\frac{8}{9}\right) \]
   \[ r = \frac{\ln\left(\frac{8}{9}\right)}{2} \approx -0.0589 \]

   Now, we can write the formula for N: \[ N(t) = 900,000 e^{-0.0589t}. \]

   Therefore, in 2007 (t = 4) the population will be
   \[ N = 900,000 e^{-0.0589 \times 4} \approx 711,087 \]

We say that a quantity A decreases (decays) exponentially with time t if \[ A(t) = A_0 e^{rt}, \] where r is a negative constant, called the rate of decay, and \( A_0 \) is the initial value of A at time t = 0.

Radioactive materials decay according to that formula, that is, the amount A of radioactive material present at time t is given by \[ A(t) = A_0 e^{rt}, \] where \( A_0 \) is the initial amount.

The graph of radioactive decay is below

Remark: Notice that the radioactive substance will never disappear completely.

The half-life of a radioactive substance is the time needed for a half of the substance to decay. For example, the half-life of radium is 1690 years, which means that if 10 grams are present today, then we have to wait 1690 years to have 5 grams (half of 10g) and another 1690 years to have 2.5 grams(half of 5g).

Example: The half-life of radioactive potassium is 1.3 billion years. If 10 grams are present now, how much will be present in 100 years?

\[
\begin{align*}
\text{Given:} & \\
A_0 &= 10 \\
t &= 100 \text{ (years)} \\
\text{half-life} &= 1.3 \times 10^9 \\
\text{Formula:} & \quad A = A_0 e^{rt}. \text{ After substituting given values, we get } A = 10 e^{100r}. \text{ Hence, to be able to compute } A, \text{ we need the rate of decay, } r. \\
\text{We use the half-life definition, to find } r. \text{ If } A_0 \text{ is the initial amount of potassium, then } A = \frac{1}{2} A_0 \text{ when } t = 1.3 \times 10^9 (1.3 \text{ billion years}). \text{ Therefore, we have (using formula } A(t) = A_0 e^{rt}): \\
\end{align*}
\]
\begin{align*}
\frac{1}{2} A_0 &= A_0 e^{-1.3 \times 10^9 r} \\
\frac{1}{2} &= e^{1.3 \times 10^9 r} \\
\ln(1/2) &= 1.3 \times 10^9 r \\
r &= \frac{\ln(1/2)}{1.3 \times 10^9} \approx -0.0053 \times 10^{-9} = -5.33 \times 10^{-10} \\

\text{Returning to the problem, we use the } r \text{ in the formula } A = 10 e^{100r} \text{ and we get} \\
A &= 10 e^{100(-5.33 \times 10^{-10})} = 10 e^{-5.33 \times 10^{-8}} \approx 9.999999467 \text{ grams.}
\end{align*}